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Summary of the PhD Thesis

Problems of regularity of systems of differential equations linearized in neighborhood of a torus

The works of A. M. Samoilenko from 1970 [5, 6] initiated new research on perturbation theory and the stability of invariant manifolds of a linear extension of a dynamical system on a $m-$dimensional torus. In these papers the Green-Samoilenko function was introduced for the first time. This contributed to the introduction of concepts such as a regular systems, a weak-regular system or a sharp-weak regular system. The linear extensions of dynamical systems on a torus are systems of the form:

$$\begin{cases} \frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx}{dt} = A(\varphi)x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $\varphi \in T_m$[1], the vector function $a(\varphi)$ is continuous and 2π-periodic with respect to each variable and additionally it satisfies the Lipschitz condition, $A(\varphi)$ is a $n \times n$-dimensional matrix whose elements are continuous, 2π-periodic with respect to each variable functions. Together with the above system we often consider the conjugated system with respect to the normal variable:

$$\begin{cases} \frac{d\varphi}{dt} = a(\varphi), \\ \frac{dy}{dt} = -A^T(\varphi)y, \end{cases} \tag{2}$$

The system (1) is regular, which means that it has exactly one Green-Samoilenko function, if there exists a function [1]:

$$V(\varphi, y) = \langle S(\varphi)y, y\rangle \,, \tag{3}$$

where $S(\varphi) \in C^1(T_m)$[2], whose derivative along the solutions of the system (2) is positive definite[3]. Since the determinant of $S(\varphi)$ is equal 0 for some $\varphi_0 \in T_m$, then system (1) is sharp-weak regular, which means that it has many different Green-Samoilenko functions, while the conjugate system will have none such function. The function (3) is often called the generalized Lyapunov function.

The goals of this work were to develop new methods for selecting the generalized Lyapunov function for certain classes of linear extensions of dynamical systems on a torus, as well as to obtain methods for

[1] T_m is called the m-dimensional torus.

[2] $S(\varphi)$ is a non-singular $n \times n$-dimensional matrix, whose elements are continuously differentiable, 2π-periodic with respect to each variable functions.

[3] The regularity of system (1) is equivalent to the regularity of system (2)

complementation process of weakly regular linear systems to regular systems.

The first chapter is about the linear extensions of dynamical systems on a torus. It starts with a theoretical introduction in which were presented the most important definitions and informations needed to understand the obtained results. The second part of the chapter presents the results of works [2], [3], during which it was possible to isolate certain classes of regular systems, for which were presented the methods of selection of the generalized Lyapunov function. During the research we found the systems of differential equations which maintain regularity under any phase perturbations. We also distinguished classes of regular systems with singular coefficient matrices.

The second chapter is devoted to the complementation process of weakly regular systems to regular systems. It starts with a theoretical introduction that includes numerous examples showing different approaches to studying the regularity of linear systems. The second part of the chapter presents the results [4] obtained in the research on complementation process of weakly regular linear systems to regular systems.

The third chapter deals with the existence of solutions to certain classes of differential inequalities. It was shown how the theory of regularity of linear extensions of dynamical systems on a torus can be used for this purpose.

References

[1] Mitropolskii Y. A., Samoilenko A. M., Kulik V. L., *Issliedowania dichotomii liniejnych sistem differencjalnych urawnienij s pomoszcziu funkcji Lapunowa*, Naukowa Dumka, Kijów, 1990.

[2] Morawiak M., *Some classes of linear extensions of dynamical system on a torus*, Mathematica Applicanda, vol. 45 iss. 2, str. 153-161, 2017.

[3] Morawiak M., *Regular classes of linear extensions of dynamical systems on a torus*, Silesian Journal of Pure and Applied Mathematics, vol. 8 iss. 1, str.21-31, 2018.

[4] Morawiak M., *On regularity of linear systems of differential equations*, Silesian Journal of Pure and Applied Mathematics, vol. 8 iss. 1, str. 5-20, 2018.

[5] Samoilenko A. M., *O sochranienii inwariantnego tora pri wozmuszczienijach*, Izw. AN SSSR Sier. mat., 34(6):1219-1240,1970.

[6] Samoilenko A. M., *On perturbation theory of invariant manifolds of dynamical systems*, Inst. Mat. Akad. Nauk Ukrainy SSR, Kijów, str 495-499, 1970.