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CHISOM PRINCE OKEKE

SYMBOLIC COMPUTATION ON A NEW CLASS OF FUNCTIONAL EQUATIONS

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Abstract

The first part of this dissertation is focused on the generalization or extension of the results of W. Fechner and E. Gselmann (Publ. Math. Debrecen 80/1-2 (2012), 143-154, https://doi.org/10.5486/PMD.2012.4970) into new classes of functional equations and determining their solutions. Since no regularity is assumed, it turns out that under some mild assumptions on the parameters involved, the pairs of functions satisfying the new classes of functional equations are polynomial functions and, in some crucial cases, just the usual polynomials. The idea to study these generalized equations was motivated by the growing number of its particular forms studied by some mathematicians.

Several mathematicians attempted to solve equations characterizing the polynomial functions of restricted domains; therefore, in this spirit, this dissertation's second part focuses on finding the local polynomial functions stemming from these new classes of functional equations.

Finally, we develop a robust computer code based on the obtained theoretical results to determine the polynomial solutions of these generalized equations. The primary motivation for writing such a computer code is that solving even simple equations belonging to these classes needs long and tiresome calculations. Therefore, one of the advantages of such a computer code is that it allows us to solve complicated problems quickly, easily, and efficiently. Additionally, the computer code will significantly improve the level of accuracy in calculations. Along with that, there is also the factor of speed. We point out that the computer code will operate with symbolic calculations provided by the Python programming language, which means that it does not contain any numerical or approximate methods, and it yields the exact solutions of the equations considered. We acknowledge that some mathematicians have previously considered using computer codes to solve functional equations. However, In their works, they used Maple as the programming tool to obtain their results which is less flexible in usage and constitutes only a small portion of the academic research community; whereas, in our work, we achieved our results using Python programming language, designed to be an easily readable, highly versatile, general-purpose, open-source, avails robustness and facilitates the deployment of theorems into computational and symbolic frameworks.

Keywords: Functional equations, Polynomial functions, Absolutely convex sets, Algebraic interior, Fréchet operator, Monomial functions, Continuity of monomial functions, Computer assisted methods, Python, Sagemath.

Streszczenie

Pierwsza część rozprawy koncentruje się na uogólnieniu lub rozszerzeniu wyników W. Fechnera i E. Gselmann (Publ. Math. Debrecen 80/1-2 (2012), 143-154, https://doi.org/10.5486/PMD.2012.4970) na nowe klasy równań funkcyjnych i wyznaczanie ich rozwiązań. Ponieważ nie zakłada się żadnej regularności, okazuje się, że przy pewnych łagodnych założeniach dotyczących parametrów pary funkcji spełniające nowe klasy równań funkcyjnych są funkcjami wielomianowymi, a w niekórych kluczowych przypadkach zwykłymi wielomianami. Pomysł zbadania tych uogólnionych równań był motywowany rosnącą liczbą jego poszczególnych form badanych przez niektórych matematyków.

Kilkoro matematyków próbowało rozwiązać równania charakteryzujące funkcje wielomianowe na dziedzinach ograniczonych; dlatego w tym duchu druga część rozprawy koncentruje się na znalezieniu lokalnych funkcji wielomianowych wynikających z tych nowych klas równań funkcyjnych.

Na koniec opracowujemy solidny kod komputerowy oparty na uzyskanych wynikach teoretycznych w celu określenia wielomianowych rozwiązań tych uogólnionych równań. Główną motywacją do napisania takiego kodu komputerowego jest to, że rozwiązanie nawet prostych równań należących do tych klas wymaga długich i męczących obliczeń. Dlatego jedną z zalet takiego kodu komputerowego jest to, że pozwala nam szybko, łatwo i skutecznie rozwiązywać skomplikowane problemy. Dodatkowo kod komputerowy znacznie poprawia poziom dokładności obliczeń. Do tego dochodzi czynnik prędkości. Zwracamy uwagę, że kod komputerowy będzie operował obliczeniami symbolicznymi zapewnianymi przez język programowania Python, co oznacza, że nie zawiera żadnych metod numerycznych ani przybliżonych i daje dokładne rozwiązania rozważanych równań. Przyznajemy, że niektórzy matematycy rozważali wcześniej użycie kodów komputerowych do rozwiązywania równań funkcyjnych. Jednak w swoich pracach wykorzystali Maple jako narzędzie programistyczne do uzyskania wyników. Maple jest mniej elastyczny w użyciu i stanowi tylko niewielką część akademickiej społeczności naukowej. Mając na uwadze, że w naszej pracy osiągnęliśmy nasze wyniki przy użyciu języka programowania Python, zaprojektowanego tak, aby był łatwo czytelnym, wysoce wszechstronnym, otwartym kodem źródłowym ogólnego przeznaczenia, zapewniającym solidność i ułatwiającym wdrażanie twierdzeń w kontekście obliczeniowym i symbolicznym.

Słowa kluczowe: Równania funkcyjne, Funkcje wielomianowe, Zbiory absolutnie wypukłe, Wnętrze algebraiczne, Operator Frécheta, Funkcje jednomianowe, Ciągłość funkcji jednomianowych, Wspomaganie komputerowe metody, Python, Sagemath.

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Contents

Chapter 1

Introduction

The classical result of L. Székelyhidi states that (under some assumptions) every solution of a general linear equation must be a polynomial function. It is known that Székelyhidi's result may be generalized to equations where some occurrences of the unknown functions are multiplied by a linear combination of the variables. In this dissertation, we study the equations where two such combinations appear. The simplest nontrivial example of such a case is given by the equation

$$F(x+y) - F(x) - F(y) = xf(y) + yf(x). \tag{1.0.1}$$

considered by W. Fechner and E. Gselmann in [11]. This dissertation is inspired by equation (1.0.1), where the first part of this dissertation is devoted mainly to finding the solutions to the various generalized forms of equation (1.0.1); that is, we find the solutions of the following functional equations;

$$\sum_{i=1}^{n} \gamma_i F(a_i x + b_i y) = yf(x) + xf(y), \tag{1.0.2}$$

$$F(x+y) - F(x) - F(y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y), \tag{1.0.3}$$

and

$$\sum_{i=1}^{n} \gamma_i F(a_i x + b_i y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y), \tag{1.0.4}$$

for every $x, y \in \mathbb{R}$, $\gamma_i, \alpha_j, \beta_j \in \mathbb{R}$, $a_i, b_i, c_j, d_j \in \mathbb{Q}$, and its special forms. Despite the fact that no regularity is assumed, it turns out that under some mild assumptions on the parameters involved, the pair (F, f) solving equations (1.0.2), (1.0.3) and (1.0.4) happens to be a pair of polynomial functions and in some crucial cases just the usual polynomials. The idea to study these generalized equations was motivated by the growing number of its particular forms studied by several mathematicians; let us quote here a few of them J. Aczél [1], J. Aczél and M. Kuczma [2], C. Alsina, M. Sablik, and J. Sikorska [4], W. Fechner and E. Gselmann [11], B. Koclęga-Kulpa, T. Szostok and S. Wąsowicz [18], [19] and [20]. From their studies, it turns out that these particular forms have real applications. The first of the special forms of (1.0.4) we solved is the functional equation considered by B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [19] namely,

$$F(x) - F(y) = (x-y)[\alpha_1 f(c_1 x + d_1 y) + \cdots + \alpha_m f(c_m x + d_m y)]. \tag{1.0.5}$$

It is worth noting that (1.0.5) stems from a well known quadrature rule used in numerical analysis. Further, we also considered other special forms of (1.0.4) namely,

$$F(y) - F(x) = \frac{1}{y-x}\int_x^y f(t)\,dt = (y-x)\sum_{j=1}^{m} \beta_j f(c_j x + (1-c_j)y), \tag{1.0.6}$$

$$F(y) - F(x) = (y - x)f(x + y), \tag{1.0.7}$$

$$F(x) - F(y) = (x - y)f\left(\tfrac{x+y}{2}\right), \tag{1.0.8}$$

and

$$2F(y) - 2F(x) = (y - x)\left(f\left(\tfrac{x+y}{2}\right) + \tfrac{f(x)+f(y)}{2}\right). \tag{1.0.9}$$

Equation (1.0.6) is the functional equation connected with the Hermite-Hadamard inequality in the class of continuous functions, and it is related to the approximate integration. Note that the quadrature rules of an approximate integration can be obtained by the appropriate specification of the coefficients of (1.0.6). Moreso, equations (1.0.7) and (1.0.8) are the functional equations considered by J. Aczél in [1] and J. Aczél and M. Kuczma in [2] respectively, which are variations of Lagrange mean value theorem with many applications in mathematical analysis, computational mathematics, and other fields. Equation (1.0.9) stems from the descriptive geometry problem considered by C. Alsina, M. Sablik, and J. Sikorska in [4] used for graphical constructions. Now observe that equations (1.0.2), (1.0.3), and (1.0.4) are an obvious generalization of the equation considered by W. Fechner and E. Gselmann in [11] given by equation (1.0.1).

Several mathematicians attempted to solve equations characterizing the polynomial functions of restricted domains, see, e.g., J. Ger ([13]) or Z. Daróczy and Gy. Maksa ([9]), therefore in this spirit, the second part of this dissertation is focused on finding the local polynomial functions stemming from equations (1.0.1) and (1.0.4), namely;

$$\tilde{F}\left(\frac{x+y}{2}\right) - \tilde{F}\left(\frac{x}{2}\right) - \tilde{F}\left(\frac{y}{2}\right) = xf(y) + yf(x), \tag{1.0.10}$$

and

$$\sum_{i=1}^{n} \gamma_i F(\lambda_i x + (1 - \lambda_i)y) = \sum_{j=1}^{m} \theta_j(\alpha_j x + (1 - \alpha_j)y)f(\beta_j x + (1 - \beta_j)y), \tag{1.0.11}$$

for every $x, y \in \mathcal{K} \subset X$–a linear space over a field $\mathbb{K} \subset \mathbb{R}$, $\mathcal{K}$ being a $\mathbb{K}$-convex subset of X, $\gamma_i, \theta_j \in \mathbb{K}$, and $\lambda_i, \alpha_j, \beta_j, \in [0, 1] \cap \mathbb{K}, i = 1, \cdots, n, j = 1, \cdots, m$. In addition, we would use our method to solve the functional equation considered by J Ger in [13], in particular

$$f(x) - f(y) = (x - y)\left[h\left(\frac{x+y}{2}\right) + k(x) + k(y)\right] \tag{1.0.12}$$

where $f, h, k : I \to \mathbb{R}$ are unknown and $I \subset \mathbb{R}$ is a non-empty interval, for all $x, y \in I$. Finally, we will develop a robust computer code based on the obtained theoretical results to determine the polynomial solutions of equation (1.0.4) and its special forms. The primary motivation for writing such a computer code is that solving even simple equations belonging to class (1.0.4) needs long and tiresome calculations. Therefore, one of the advantages of such a computer code is that it allows us to solve complicated problems quickly, easily, and efficiently. Additionally, the computer code will significantly improve the level of accuracy in calculations. Along with that, there is also the factor of speed. We point out that the computer code will operate with symbolic calculations provided by Python programming language, which means that it does not contain any numerical or approximate methods, and it yields the exact solutions of the equations considered. We acknowledge that some mathematicians have previously considered using computer codes to solve functional equations. We mention here some of them, S. Baják and Z. Páles [5], and [6], G.G. Borus and A. Gilányi [7], A. Házy [16], and [17], and C.P. Okeke and M. Sablik [28]. In their works, they used Maple as the programming tool to obtain their results which is less flexible in usage and constitutes only a small portion of the academic research community; however, in our work, we achieved our results using Python programming language, designed to be an easily readable, highly versatile, general-purpose, open-source, avails robustness and facilitates the deployment of theorems into computational and symbolic frameworks.

List of publications

The results presented in this dissertation are solely based on the following publications:

(1) T. Nadhomi, **C. P. Okeke**, M. Sablik and T. Szostok, On a new class of functional equations satisfied by polynomial functions. *Aequationes Math., 95* (2021), 1095-1117.

(2) **C. P. Okeke**, M. Sablik, Functional equation characterizing polynomial functions and an algorithm. Results Math 77, 125 (2022).

(3) **C. P. Okeke**, Further results on a new class of functional equations satisfied by polynomial functions. Results Math 78, 96 (2023). https://doi.org/10.1007/s00025-023-01877-8

(4) **C. P. Okeke**, W. Ogala and T. Nadhomi, On symbolic computation of C.P. Okeke functional equations using python programming language. Submitted.

(5) **C. P. Okeke**, M. Sablik, Characterizing locally polynomial functions on convex subsets of linear spaces. Submitted.

Chapter 2

Preliminaries

In this chapter, we begin by presenting some basic definitions.

2.1 Polynomial functions

The history of polynomial functions goes back to the year 1909 when the paper by M. Fréchet [12] appeared. Let G, H be abelian groups (for some results concerning the noncommutative case see the papers of J. Almira and E. Shulman [3] and E. Shulman [34]) and let $f : G \to H$ be a given function. The Fréchet operator (difference operator) Δ_h with span $h \in G$ is defined by

$$\Delta_h f(x) := f(x+h) - f(x)$$

and Δ_h^n is defined recursively

$$\Delta_h^0 f := f, \ \Delta_h^{n+1} f := \Delta_h(\Delta_h^n f) = \Delta_h \circ \Delta_h^n f, \ n \in \mathbb{N}.$$

Using this operator, polynomial functions are defined in the following way.

Definition 2.1.1. *Fix a nonnegative integer n, and let $(G, +)$ and $(H, +)$ be groups. We say that a function $f : G \to H$ satisfies the* Fréchet equation *(of order n) if, and only if*

$$\Delta_{y_{n+1},\dots,y_1}^{n+1} f(x) = 0, \tag{2.1.1}$$

for all $y_1, \dots, y_{n+1}, x \in G$. We say that any solution to (2.1.1) is a polynomial function of order at most n.

Polynomial functions are sometimes called *generalized polynomials*. The shape of solutions of this equation was obtained in various situations among others by S. Mazur and W. Orlicz [23], G. Van der Lijn [37] and D. Z. Đoković [10]. To describe the form of polynomial functions we need the notion of multiadditive functions. A function $A_n : G^n \to H$ is $n-$additive if and only if for every $i \in \{1, 2, \dots, n\}$ and for all $x_1, \dots, x_n, y_i \in G$ we have

$$\begin{aligned} A_n(x_1, \dots, &x_{i-1}, x_i + y_i, x_{i+1}, \dots, x_n) \\ &= A_n(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) + A_n(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n). \end{aligned}$$

Further, for a function $A_n : G^n \to H$, the diagonalization A_n^* is defined by

$$A_n^*(x) := A_n\underbrace{(x, \cdots, x)}_{n \text{ times}}.$$

Definition 2.1.2. *A function $A : G \longrightarrow H$ is called additive if and only if it satisfies the Cauchy's functional equation i.e.*

$$A(x + y) = A(x) + A(y) \tag{2.1.2}$$

for all $x, y \in G$.

Theorem 2.1.1. *If $A : G \longrightarrow H$ satisfies (2.1.2), then $A(\lambda x) = \lambda A(x)$* for every x ∈ G and $\lambda \in \mathbb{Q}$.

Proof. For $x = y = 0$ we get from (2.1.2), $A(0) = 0$. Now we show by induction that $A(\lambda x) = \lambda A(x)$ for $\lambda \in \mathbb{N}$, $x \in G$. Obviously the above formula holds for $\lambda = 1$. Assume that (2.1.2) holds for some $\lambda \in \mathbb{N}$ then we have from induction assumption and additivity of A that,

$$\begin{aligned} A((\lambda + 1)x) &= A(x + \lambda x) = A(x) + A(\lambda x) \\ &= A(x) + \lambda A(x) = (\lambda + 1)A(x) \end{aligned}$$

whence, setting in (2.1.2) $y = -x$, we obtain

$$\begin{aligned} 0 = A(0) = A(x - x) &= A(x) + A(-x) \\ A(-x) &= -A(x) \end{aligned}$$

Therefore we've, $A(\lambda x) = \lambda A(x)$ for $\lambda \in \mathbb{Z}$, $x \in G$. Now, since an arbitrary $\lambda \in \mathbb{Q}$ may be written as $\lambda = \frac{p}{q}$,where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, hence $px = q(\lambda x)$ and so, by what already has been proved,

$$pA(x) = A(px) = A(q(\lambda x)) = qA(\lambda x),$$

so,

$$\tfrac{p}{q}A(x) = A(\lambda x)$$

hence,

$$\lambda A(x) = \tfrac{p}{q}A(x) = A(\lambda x)$$

for every $x \in G$ *and* $\lambda \in \mathbb{Q}$. □

Remark 2.1.1. *Take $G = H = \mathbb{R}$, any continuous additive function $A : \mathbb{R} \to \mathbb{R}$ is of the form*

$$A(x) = cx, \quad x \in \mathbb{R} \tag{2.1.3}$$

where $c \in \mathbb{R}$ is a constant.

Now, we present the characterization of polynomial functions.

Theorem 2.1.2. *(cf. Theorem 9.1 in [35]) Let $(G, +)$ be a commutative semigroup with identity, let $(H, +)$ be a commutative group and let n be a nonnegative integer. Moreover, assume that H is uniquely divisible by $n!$. Then $f : G \to H$ is a solution of (2.1.1) if and only if it has the form*

$$f(x) = \sum_{k=0}^{n} A_k^*(x), \tag{2.1.4}$$

for all $x \in G$ where A_k^*, $k \in \{0, \cdots, n\}$, are diagonalizations of k-additive symmetric functions $A_k : G^k \to H$.

In the case of $G = H = \mathbb{R}$, we obtain the following

Corollary 2.1.1. *(cf. Corollary 1.1 in [27] and [29]) Let n be a nonnegative integer. Then $f : \mathbb{R} \to \mathbb{R}$ is a* continuous *solution of (2.1.1) if and only if it has the form*

$$f(x) = \sum_{k=0}^{n} a_k x^k, \tag{2.1.5}$$

where a_k, $k \in \{0, \cdots, n\}$, are some real constants.

In other words, a continuous real solution of (2.1.1) is an ordinary polynomial. Next, we define a polynomial function on a restricted domain.

2.2 Locally polynomial functions

Definition 2.2.1. *(cf. Definition 1.1 in [29]) Suppose that X, Y are linear spaces over a field $\mathbb{K} \subset \mathbb{R}$, let $\mathcal{K}$ be a non-empty subset of X. A map $f : \mathcal{K} \longrightarrow Y$ is called a locally polynomial function of degree at most n on $\mathcal{K}$, if*

$$\Delta^{n+1}_{y_1,\dots,y_{n+1}} f(x) = 0,$$

holds for every $x, y_i \in X$ such that $x + \sum_{i\in Z} y_i \in \mathcal{K}$, $Z \subset \{1, \cdots, n+1\}$.

We will work on absolutely convex (convex and balanced) sets with non-empty algebraic interiors to obtain the local solutions. Let us remind the definition of an algebraic interior and absolutely convex sets.

Definition 2.2.2. *(cf. Definition 1.2 in [29]) Let $\mathcal{K} \subset X$ -a linear space over a field $\mathbb{K} \subset \mathbb{R}$. The algebraic interior of the set $\mathcal{K}$ is the set*

$$\mathrm{algint}\mathcal{K} = \{y \in \mathcal{K} : \bigwedge_{x\in X} \bigvee_{\varepsilon>0} \bigwedge_{\alpha\in(-\varepsilon,\varepsilon)\cap\mathbb{K}} (\alpha x + y \in \mathcal{K})\}$$

Definition 2.2.3. *(cf. Definition 1.3 in [29]) Let $\mathcal{K} \subset X$ -a linear space over a field $\mathbb{K} \subset \mathbb{R}$. The set $\mathcal{K}$ is said to be* absolutely convex *if it is convex and balanced, i.e.*

$$\bigwedge_{\alpha\in\mathbb{K}} (|\alpha| \leqslant 1 \Longrightarrow \alpha\mathcal{K} \subset \mathcal{K}).$$

Now, let us present a lemma which follows easily from the well known properties of absolutely convex sets (cf. [21] and [40]). We admit the following definitions

(i) $I := \{(\alpha, \beta) \in \mathbb{K} \times \mathbb{K} : |\alpha| + |\beta| \leqslant 1\}$,

(ii) $I^0 := \{(\alpha, \beta) \in I : \beta \neq 0\}$.

Lemma 2.2.1. *(cf. Lemma 1.3 in [30]) Let $\emptyset \neq \mathcal{K} \subset X$ -a linear space over a field $\mathbb{K} \subset \mathbb{R}$, be absolutely convex and suppose that $J \subset I$ is finite. Further, let $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n) \in I^0$ for a fixed $n \in \mathbb{N}$. If*

$$r \geqslant \max\left\{n+1, \max\left\{|\alpha| + |\beta| + \sum_{i=1}^{n} \left|\frac{\alpha\beta_i - \alpha_i\beta}{\beta_i}\right| : (\alpha, \beta) \in J\right\}\right\}, \tag{2.2.1}$$

then for every $x, y, u_1, \cdots, u_n \in \frac{1}{r}\mathcal{K}$, every $(\alpha, \beta) \in J$, and $S \subset \{1, \cdots, n\}$ we have

(a) $x + \sum_{i\in S} u_i \in \mathcal{K}$,

(b) $\alpha x + \beta y + \sum_{i\in S} \frac{\alpha\beta_i - \alpha_i\beta}{\beta_i} u_i \in \mathcal{K}$.

We quote here a convex version of Lemma 2.1 in [31], which is used in the proof of our main result obtained in Chapter 6 (cf. [29]).

Lemma 2.2.2. *(cf. Lemma 1.4 in [30]) Fix $N \in \mathbb{N}$. Suppose that $\mathcal{K}$ is a non-empty convex set of X -a linear space over a field $\mathbb{K} \subset \mathbb{R}$, such that $0 \in algint\mathcal{K}$, and G a group uniquely divisible by $N!$. Let $B_i : \mathcal{K} \to G$, $i \in \{0, \cdots, N\}$ be functions homogeneous of the i^{th} order with respect to $\{2, \cdots, N+1\}$, i.e. satisfying for every $i \in \{0, \cdots, N\}$, every $k \in \{2, \cdots, N+1\}$ and every $z \in \frac{1}{N+1}\mathcal{K}$*

$$B_i(kz) = k^i B_i(z).$$

If $B_N(x) + \cdots + B_0(x) = 0$, for every $x \in \mathcal{K}$, then $B_N = \cdots = B_0 = 0$.

Let us mention a very important result used in [25], [28], [26] and [27] due to L. Székelyhidi who proved that every solution of a very general linear equation is a polynomial function (see [35] Theorem 9.5, cf. also W. H. Wilson [39]).

Theorem 2.2.1. *(Theorem 1.2 in [26]) Let G be an Abelian semigroup, S an Abelian group, n a positive integer, φ_i, ψ_i additive functions from G to G and let $\varphi_i(G) \subset \psi_i(G)$, $i \in \{1, \dots, n\}$. If functions $f, f_i : G \to S$ satisfy equation*

$$f(x) + \sum_{i=1}^{n} f_i(\varphi_i(x) + \psi_i(y)) = 0, \tag{2.2.2}$$

then f satisfies (2.1.1).

The Székelyhidi's result makes it easier to solve linear equations because it is no longer necessary to deal with each equation separately. Instead, we may formulate results which are valid for large classes of equations.

In subsequent chapters we will present our results obtained in [25], [28], [26], [27] and [29]. Before we state the results, let us adopt the following notation. Let G and H be a commutative groups. Then $SA^i(G; H)$ denotes the group of all i-additive, symmetric mappings from G^i into H for $i \geqslant 2$, while $SA^0(G; H)$ denotes the family of constant functions from G to H and $SA^1(G; H) = Hom(G; H)$. We also denote by $\mathcal{I}$ the subset of $Hom(G; G) \times Hom(G; G)$ containing all pairs (α, β) for which $Ran(\alpha) \subset Ran(\beta)$. Furthermore, we adopt a convention that a sum over an empty set of indices equals zero. We denote also for an $A_i \in SA^i(G; H)$ by A_i^* the diagonalization of A_i, $i \in \mathbb{N} \cup \{0\}$. We denote also for an $A_i \in SA^i(G; H)$ by A_i^* the diagonalization of A_i, $i \in \mathbb{N} \cup \{0\}$. Let us also introduce the operator $\Gamma : G \times G \times H^{G \times G} \to H^{G \times G}$ defined as follows. For each $\phi : G \times G \to H$ and each $(u, v) \in G \times G$ we set

$$\Gamma_{(u,v)}\phi(x, y) := \phi(x + u, y + v) - \phi(x, y),$$

for each $(x, y) \in G \times G$. In fact, Γ is nothing else but the operator Δ defined above applied to functions of two variables. However we wish to stress the difference between one and two variables, this is why we denote the new operator with a different symbol.

Chapter 3

On a new class of functional equations satisfied by polynomial functions

Here we present the result obtained in [25] where we generalized the left-hand side of the Fechner-Gselmann equation given by equation (1.0.2).

The fundamental tool in achieving the results in [25], [28], [26] and [27] is a very special Lemma (cf. Lemma 2.1 in [25], Lemma 1.1 in [28], Lemma 2.1 in [26] and Lemma 1.1 in [27]). Let us observe that this is the modified version of Lemma 1 in [22] and Lemma 2.3 in [31]. The result is a generalization of theorems from L. Székelyhidi's book [35] (Theorem 9.5), which in turn is a generalization of a W. H. Wilson result from [39].

3.1 Sablik Lemma

Lemma 3.1.1. *(cf. Lemma 2.1 in [26]) Fix $N \in \mathbb{N} \cup \{0\}$, $M \in \mathbb{N} \cup \{-1, 0\}$ and, if $M \geqslant 0$, let $I_{p,n-p}$, $0 \leqslant p \leqslant n$, $n \in \{0, \dots, M\}$ be finite subsets of $\mathcal{I}$. Suppose further that H is an Abelian group uniquely divisible by $N!$ and G is an Abelian group. Moreover, let functions $\varphi_i : G \to SA^i(G; H)$, $i \in \{0, \dots, N\}$ and, if $M \geqslant 0$, $\psi_{p,n-p,(\alpha,\beta)} : G \to SA^i(G; H)$, $(\alpha, \beta) \in I_{p,n-p}$, $0 \leqslant p \leqslant n, n \in \{0, \dots, M\}$,satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = R_M(x, y), \tag{3.1.1}$$

where $R_M(x, y)$ is defined in the following way

$$R_M(x,y) = \begin{cases} 0, & M = -1, \\ \sum\limits_{n=0}^{M} \sum\limits_{p=0}^{n} \sum\limits_{(\alpha,\beta)\in I_{p,n-p}} \psi_{p,n-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^p, y^{n-p}), & M \geqslant 0 \end{cases}$$

for every $x, y \in G$. Then φ_N is a polynomial function of degree not greater than m, where

$$m = \sum_{n=0}^{M} \operatorname{card} \left(\bigcup_{s=n}^{M} K_s \right) - 1, \tag{3.1.2}$$

and $K_s = \bigcup_{p=0}^{s} I_{p,s-p}$ for each $s \in \{0, \dots, M\}$, if $M \geqslant 0$. Moreover, if $M = -1$,

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = 0$$

then $m = -1$ and φ_N is the zero function.

Proof. Let us fix an $N \in \mathbb{N} \cup \{0\}$. We prove the Lemma using induction with respect to M. Let us start with $M = -1$; we mean that the right-hand side of the equation (3.1.1) vanishes. Thus (3.1.1) reduces to

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = 0, \tag{3.1.3}$$

for each x, $y \in G$. It turns out that the polynomial in y and coefficients $\varphi_i(x)$, $i \in \{0, \dots, N\}$ vanishes identically. It is not difficult to see that it is equivalent to the system of identities $\varphi_i = 0$, $i \in \{0, \dots, N\}$. In particular φ_N is a polynomial function, identically equal to 0, the degree is hence estimated by 0.

Now suppose that our Lemma holds for some $M > -1$ and consider the equation

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{n=0}^{M+1} \sum_{p=0}^{n} \sum_{(\alpha,\beta)\in I_{p,n-p}} \psi_{p,n-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^p, y^{n-p}) \tag{3.1.4}$$

for every $x, y \in G$. Assume that $K_{M+1} \neq \emptyset$ - otherwise (3.1.4) reduces to (3.1.1) and we are done. Further, assume that $I_{p,M+1-p} \neq \emptyset$ for some $p \in \{0, \dots, M+1\}$. Fix such a p and write $I_{p,M+1-p} = \{(\alpha_j, \beta_j) : j \in \{1, \dots, m\}\}$ for some $m \in \mathbb{N}$. Choose a pair $(\alpha, \beta) \in I_{p,M+1-p}$ and fix a $u_1 \in G$ arbitrarily. To the u_1 take a $v_1 \in \beta^{-1}(\{\alpha(-u_1)\})$ so that $\alpha(u_1) + \beta(v_1) = 0$. Now let us apply the operator $\Gamma_{(u_1,v_1)}$ to both sides of (3.1.4). On the left-hand side we obtain

$$\begin{aligned}
&\varphi_N(x+u_1)((y+v_1)^N) - \varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \Gamma_{(u_1,v_1)}\varphi_i(x)(y^i) \\
&\quad = \varphi_N(x+u_1)(y^N) - \varphi_N(x)(y^N) + \sum_{k=1}^{N} \binom{N}{k} \varphi_N(x+u_1)(y^{N-k}, v_1^k) \\
&\quad + \sum_{i=0}^{N-1} \Gamma_{(u_1,v_1)}\varphi_i(x)(y^i) = \Delta_{u_1}\varphi_N(x)(y^N) + \sum_{k=1}^{N} \binom{N}{k} \Delta_{u_1}\varphi_N(x)(y^{N-k}, v_1^k) \\
&\quad + \sum_{k=1}^{N} \binom{N}{k} \varphi_N(x)(y^{N-k}, v_1^k) + \sum_{i=0}^{N-1} \Gamma_{(u_1,v_1)}\varphi_i(x)(y^i).
\end{aligned} \tag{3.1.5}$$

Denoting $\hat{\varphi}_N := \Delta_{u_1}\varphi_N$ we get again the right-hand side of equation (3.1.1) but with $\hat{\varphi}_N$ instead of φ_N (note that the remaining summands may be written as polynomial functions in y but of degrees lower than N, and they can be rearranged in such a way that the left-hand side is again a finite sum of polynomial functions in y with coefficients dependent on x).

Let us look now at the right-hand side. If we apply $\Gamma_{(u_1,v_1)}$ to the first summands it will transform them into summands of similar character, with $\alpha(x)+\beta(y)$ replaced by $\alpha(x)+\beta(y)+\alpha(u_1)+\beta(v_1)$. But in the last summand, and more exactly in the summand determined by the pair (α, β) to which u_1 and v_1 were selected, we have the following situation

$$\begin{aligned}
&\psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y) + \alpha(u_1) + \beta(v_1)\right) ((x+u_1)^p, (y+v_1)^{M+1-p}) \\
&\quad -\psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^p, y^{M+1-p}) \\
&\quad = \psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) ((x+u_1)^p, (y+v_1)^{M+1-p}) \\
&\quad -\psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^p, y^{M+1-p}) \\
&\quad = \psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^p, y^{M+1-p}) \\
&\quad -\psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x)\beta(y)\right) (x^p, y^{M+1-p}) \\
&\quad + \sum_{(s,t)\in\{0,\cdots,p\}\times\{0,\cdots,M+1-p\}\setminus\{(0,0)\}} \binom{p}{s}\binom{M+1-p}{t} \psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^{p-s}, u_1^s, y^{M+1-p-t}, v_1^t) \\
&\quad = \sum_{(s,t)\in\{0,\cdots,p\}\times\{0,\cdots,M+1-p\}\setminus\{(0,0)\}} \binom{p}{s}\binom{M+1-p}{t} \psi_{p,M+1-p,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right) (x^{p-s}, u_1^s, y^{M+1-p-t}, v_1^t),
\end{aligned} \tag{3.1.6}$$

for every $x, y \in G$. We see that the action of $\Gamma_{(u_1,v_1)}$ increases the number of summands but decreases the degree of polynomial functions by 1. Applying the operator $p-1$ more times we will eventually annihilate the summand on the right-hand side. Repeating the above procedure for arbitrary $u_j \in G$, $j \in \{1, \cdots, q\}$ we obtain equation (cf. (3.1.5) and (3.1.6))

$$\begin{aligned}
&\Delta_{u_1,\ldots,u_q}\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \hat{\varphi}_i(x)(y^i) \\
&= \sum_{n=0}^{M} \sum_{p=0}^{n} \sum_{(\alpha,\beta)\in I_{p,n-p}} \hat{\psi}_{p,n-p,(\alpha,\beta)} \left(\alpha(x)+\beta(y)\right)(x^p, y^{n-p}) \\
&+ \sum_{j=0, j\neq p}^{m} \sum_{(\alpha,\beta)\in I_{j,M+1-j}} \hat{\psi}_{j,M+1-j,(\alpha,\beta)} \left(\alpha(x)+\beta(y)\right)(x^j, y^{M+1-j}),
\end{aligned} \tag{3.1.7}$$

for every $x, y \in G$. Here $\hat{\psi}_{p,n-p,(\alpha,\beta)}$ and $\hat{\varphi}_i$ are new functions obtained after applying Γ operator to the previous ones. Anyway, the method shows that repeating it we may arrive at complete annihilation of the summand corresponding to $M+1$ and finally replace (3.1.7) by the following.

$$\begin{aligned}
&\Delta_{u_1,\ldots,u_q}\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \hat{\varphi}_i(x)(y^i) \\
&= \sum_{n=0}^{M} \sum_{p=0}^{n} \sum_{(\alpha,\beta)\in I_{p,n-p}} \hat{\psi}_{p,n-p,(\alpha,\beta)} \left(\alpha(x)+\beta(y)\right)(x^p, y^{n-p}),
\end{aligned} \tag{3.1.8}$$

for all $x, y \in G$ and $u_1, \ldots, u_q \in G$. Now we may use the induction hypothesis and infer that

$$\Delta_{u_1,\ldots,u_q}\varphi_N$$

is a polynomial function.

The estimation of the degree consists in realizing what is happening indeed. Applying the operator $\Gamma_{(u,v)}$ (with properly selected u and v) to both sides we "annihilate" one summand on the right-hand side of (3.1.1) at the level 0. Thus, applying the operator Γ cardK_0 times with arbitrary u's we get rid of the summands constituting the level 0. Then we apply again Γ to annihilate the level 1 summands, but we have to do it in two steps. First we decrease the degree of summand by 1 and only then, in step two, we can annihilate the summand. It takes thus 2cardK_1 to annihilate the terms of the degree 1. Similarly, it takes 3cardK_2 to annihilate terms of the second degree, and, in general, $(n+1)$cardK_n to annihilate terms of the n-th degree. On the left-hand side appears the sign of $\Delta_{u_1,\ldots,u_q}\varphi_N(x)(y)$ where

$$q = \sum_{n=0}^{M} \operatorname{card}\left(\bigcup_{s=n}^{M} K_s\right).$$

□

With this lemma, it can be proved(under some mild assumptions) that at least one of the functions in equations (1.0.2), (1.0.3), and (1.0.4) is a polynomial function.

3.2 Results

3.2.1 Fechner-Gselmann equation

Let us solve equation (1.0.1) by applying our Lemma 3.1.1.

Theorem 3.2.1. *(cf. Theorem 3.1 in [25]) Let the pair (F, f) of functions mapping $\mathbb{R}$ to $\mathbb{R}$ satisfy the equation*

$$F(x+y) - F(x) - F(y) = yf(x) + xf(y) \tag{3.2.1}$$

for all $x, y \in \mathbb{R}$. Then f is a polynomial function of degree not greater than 2 and F is a polynomial function of degree not greater than 3.

Proof. Let us rewrite equation (3.2.1) in the form

$$f(x)y + F(x) = -f(y)x + F(x+y) - F(y) \tag{3.2.2}$$

for all $x, y \in \mathbb{R}$. If we take now $G = H = \mathbb{R}$, $N = 1$, $M = 1$, $I_{0,0} = \{(0, \mathrm{id}), (\mathrm{id}, \mathrm{id})\}$, $\psi_{0,0,(0,\mathrm{id})} = -F$, $\psi_{0,0,(\mathrm{id},\mathrm{id})} = F$, $I_{0,1} = \emptyset$, $I_{1,0} = \{(0, \mathrm{id})\}$, $\psi_{1,0,(0,\mathrm{id})} = -f$, $\varphi_1 = f$, $\varphi_0 = F$ then we see that (3.2.2) is a particular case of (3.1.1). We also have $K_0 = I_{0,0}$ and $K_1 = I_{1,0}$ with $\mathrm{card}(K_0 \cup K_1) = 2$ and $\mathrm{card} K_1 = 1$. Therefore (cf. (3.1.2)) f is a polynomial function of degree at most 2. Hence there exist $A_0 \in SA^0(\mathbb{R}, \mathbb{R})$, $A_1 \in SA^1(\mathbb{R}, \mathbb{R})$ and $A_2 \in SA^2(\mathbb{R}, \mathbb{R})$ such that f is given by

$$f(x) = A_0^* + A_1^*(x) + A_2^*(x) \tag{3.2.3}$$

for every $x \in \mathbb{R}$. On the other hand, taking (3.2.1) into consideration again and putting $y = h$ in (3.2.1) we obtain after rearranging the equation

$$F(x+h) - F(x) = hf(x) + xf(h) + F(h),$$

or

$$\Delta_h F(x) = hf(x) + xf(h) + F(h). \tag{3.2.4}$$

Since f is a polynomial function, we see that the right-hand side of the above is a polynomial function. Now, applying the Fréchet operator three times to both sides of (3.2.4) we see that the right-hand side vanishes and so does the left-hand side. This means however that F is a polynomial function of order greater by 1 than order of f. □

Remark 3.2.1. *(cf. Remark 3.1 in [25]) In fact we have shown above that the class of polynomial functions has the so called* double difference property*, more exactly if DF defined by $DF(x, y) = F(x+y) - F(x) - F(y)$ is a polynomial function of two variables then $F = a + p$, where $a : \mathbb{R} \longrightarrow \mathbb{R}$ is an additive function and $p : \mathbb{R} \longrightarrow \mathbb{R}$ is a polynomial function.*

Let $B_i \in SA^i(\mathbb{R}, \mathbb{R})$, $i \in \{0, \ldots, 3\}$ be such that

$$F(x) = B_0^* + B_1^*(x) + B_2^*(x) + B_3^*(x) \tag{3.2.5}$$

for every $x \in \mathbb{R}$.

Remark 3.2.2. *(cf. Remark 3.2 in [25]) In (3.2.1) taking $qx; qy$ in places of x and y; respectively, using the rational homogeneity of monomial summands of F and f and joining together the terms with equal powers of q we can see that this equation is possible only if it occurs for monomials of equal order.*

Taking the above remark into account, we start with $F = B_0^* = B_0$. Then from (3.2.1) we infer that $f = 0$ and so

$$-B_0 = 0,$$

In particular, $F(0) = 0$. Let us now assume that $F(x) = B_1^*(x) = B_1(x)$. Then necessarily (cf. (3.2.1))

$$0 = F(2x) - 2F(x) = 2xf(x)$$

whence it follows that $f = 0$. Thus B_1 is an arbitrary additive function, and in particular $A_0 = 0$.

The next step is

$$F(x) = B_2^*(x) = B_2(x,x)$$

for every $x \in \mathbb{R}$. From (3.2.1) we derive

$$2B_2(x,y) = xA_1(y) + yA_1(x)$$

for every $x, y \in \mathbb{R}$. Hence

$$B_2^*(x) = xA_1(x) \tag{3.2.6}$$

for every $x \in \mathbb{R}$. Now, let us pass to the case where $F(x) = B_3^*(x)$ for every $x \in \mathbb{R}$. Then we have $f(x) = A_2^*(x)$, $x \in \mathbb{R}$ and from (3.2.1) we get, taking $x = y$

$$6B_3^*(x) = 2xA_2^*(x)$$

whence

$$B_3^*(x) = \frac{1}{3}xA_2^*(x) \tag{3.2.7}$$

for every $x \in \mathbb{R}$. Inserting the above equality into (3.2.1) we obtain

$$(x+y)A_2^*(x+y) - xA_2^*(x) - yA_2^*(y) = 3\left(xA_2^*(y) + yA_2^*(x)\right)$$

for every x, $y \in \mathbb{R}$. After some elementary calculations we obtain hence

$$(x+y)A_2(x,y) = yA_2^*(x) + xA_2^*(y)$$

for every x, $y \in \mathbb{R}$. Putting here $y = 1$ we obtain

$$xA_2(x,1) + A_2(x,1) = A_2(x,x) + xA_2^*(1) \tag{3.2.8}$$

for every $x \in \mathbb{R}$. We obtain from (3.2.8)

$$A_2(x,1) = xA_2^*(1)$$

and

$$A_2^*(x) = xA_2(x,1) = x^2A_2^*(1) \tag{3.2.9}$$

for every $x \in \mathbb{R}$. Taking into account (3.2.7) we have by (4.1.5)

$$B_3^*(x) = \frac{1}{3}x^3A_2^*(1) \tag{3.2.10}$$

for every $x \in \mathbb{R}$. Thus we have proved the following.

Proposition 3.2.1. *(cf. Proposition 3.2 in [25]) The pair (F, f) is a solution of (3.2.1) if, and only if*

- $f(x) = A_1(x) + a_2x^2$,
- $F(x) = B_1(x) + xA_1(x) + \frac{1}{3}a_2x^3$,

for all $x \in \mathbb{R}$. Here A_1 and B_1 are arbitrary additive functions, and $a_2 \in \mathbb{R}$ is an arbitrary constant.

3.2.2 Generalized left-hand side of the Fechner-Gselmann equation

Now we are going to investigate a slightly different equation. We are interested in solving the equation

$$\sum_{i=1}^{n} \gamma_i F(\alpha_i x + \beta_i y) = x f(y) + y f(x) \tag{3.2.11}$$

for every $x, y \in \mathbb{R}$. First, we assume that both functions F and f are polynomial functions. Then, similarly as in the case of Theorem 3.2.1, the monomial summands of F and f of orders $k+1$ and k, respectively satisfy (3.2.11). Later on we will discuss how Lemma 3.1.1 may be used to show that (in some situations) F and f are indeed polynomial functions.

A characteristic feature of (3.2.11) is dependence of the existence of solutions on the behaviour of the sequence $(S_k)_{k\in\mathbb{N}}$ given by

$$S_k = \sum_{i=1}^{n} \gamma_i (\alpha_i + \beta_i)^{k+1}, \tag{3.2.12}$$

for all $k \in \mathbb{N} \cup \{0\}$. Let us observe that in the case of (3.2.1) we have $n = 3$ and $\gamma_1 = \alpha_1 = \beta_1 = \alpha_2 = \beta_3 = 1$, and $\beta_2 = \alpha_3 = 0$ while $\gamma_2 = \gamma_3 = -1$. We have $S_k = 2^{k+1} - 2 = 2(2^k - 1)$, $k \in \mathbb{N}$, in particular $S_0 = 1 \cdot 2 - 1 \cdot 1 - 1 \cdot 1 = 0$.

Using our Lemma 3.1.1 we infer rather easily that f is a polynomial function. We assume that also F is a polynomial function. The aim of the next theorem is to prove that, under the assumptions made, solutions of (3.2.11) are continuous, except for an additive summand. Similarly as in the case of Theorem 3.2.1, it is enough to assume that F and f are monomials.

Theorem 3.2.2. *(cf. Theorem 3.3 in [25]) Let $k \in \mathbb{N} \cup \{0\}$. Let $\gamma_i \in \mathbb{R}$, $\alpha_i, \beta_i \in \mathbb{Q}$ be such that (cf. (3.2.12)) $S_k \neq 0$, $k \in \mathbb{N} \cup \{0\}$. Further, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be either 0 or a monomial function of order k, let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a monomial function of order $k+1$ and suppose that the pair (F, f) satisfies equation (3.2.11).*

(i) If $k = 1$ then either $\sum_{i=1}^{n} \gamma_i \alpha_i^2 \neq 0 \neq \sum_{i=1}^{n} \gamma_i \beta_i^2$ and $f = F = 0$ is the only solution of (3.2.11), or $\sum_{i=1}^{n} \gamma_i \alpha_i^2 = \sum_{i=1}^{n} \gamma_i \beta_i^2 = 0$ and f is an arbitrary additive function while F is given by $F(x) = \frac{2}{S_1} x f(x)$.

(ii) If $k = 0$ or $k \geqslant 2$ then both f and F are continuous.

Moreover, for every $k > 2$ and for every $j \in \{2, \ldots, k-1\}$, if $f \neq 0$ then

$$\sum_{i=1}^{n} \gamma_i \alpha_i^j \beta_i^{k+1-j} = 0, \tag{3.2.13}$$

$$\sum_{i=1}^{n} \gamma_i \alpha_i^{k+1} \beta_i = \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i^{k+1} = 0, \tag{3.2.14}$$

which implies

$$S_k = 2(k+1) \sum_{i=1}^{n} \gamma_i \alpha_i^k \beta_i = 2(k+1) \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i^k, \tag{3.2.15}$$

and obviously

$$\sum_{i=1}^{n} \gamma_i \alpha_i^k \beta_i = \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i^k. \tag{3.2.16}$$

Proof. Let us start with the case $k = 0$. Then $f = \text{const} = A_0$ and F is additive. Putting $x = y$ in (3.2.11) we obtain (taking into account rational homogeneity of F)

$$S_0 F(x) = 2xA_0,$$

for each $x \in \mathbb{R}$. Using the assumption that $S_0 \neq 0$ we get

$$F(x) = \frac{2}{S_0} A_0 x,$$

for each $x \in \mathbb{R}$, and hence F is a continuous function.

In the case $k = 1$ we obtain that $f = A_1$ is additive and F is a quadratic function, i.e. a diagonalization of a biadditive symmetric function, $S_1 = \sum_{i=1}^{n} \gamma_i(\alpha_i + \beta_i)^2$. Putting $x = y$ in (3.2.11) we obtain

$$S_1 F(x) = 2xA_1(x),$$

for every $x \in \mathbb{R}$, whence (keeping in mind that $S_1 \neq 0$) we get (denoting $\frac{2}{S_1}$ by C_1)

$$F(x) = C_1 x A_1(x) \tag{3.2.17}$$

for every $x \in \mathbb{R}$. Substituting the above into (3.2.11), we obtain

$$\begin{aligned} &C_1 \sum_{i=1}^{n} \gamma_i \left[(\alpha_i x + \beta_i y)\left(\alpha_i A_1(x) + \beta_i A_1(y)\right)\right] \\ &\quad = C_1 \left[\left(\sum_{i=1}^{n} \gamma_i \alpha_i^2\right) x A_1(x) + \left(\sum_{i=1}^{n} \gamma_i \beta_i^2\right) y A_1(y)\right] \\ &\quad + C_1 \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i \left(x A_1(y) + y A_1(x)\right) \\ &\quad = x A_1(y) + y A_1(x), \end{aligned} \tag{3.2.18}$$

for all x, $y \in \mathbb{R}$. Comparing terms of the same degree on both sides of the above equation, we obtain

$$\sum_{i=1}^{n} \gamma_i \alpha_i^2 x A_1(x) = 0,$$

for all $x \in \mathbb{R}$, and symmetrically,

$$\sum_{i=1}^{n} \gamma_i \beta_i^2 y A_1(y) = 0,$$

for all $y \in \mathbb{R}$. Both of these equations hold if either $A_1 = 0$ or

$$\sum_{i=1}^{n} \gamma_i \alpha_i^2 = 0 = \sum_{i=1}^{n} \gamma_i \beta_i^2 = 0. \tag{3.2.19}$$

Now if $A_1 = 0$ then also $F = 0$, and we get the continuity of a solution (F, f) of (3.2.11) in this case. Further let us look for non-zero solutions of (3.2.11). The existence of a nontrivial A_1 implies that (3.2.19) holds. So, in this case we have

$$S_1 = 2 \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i. \tag{3.2.20}$$

Taking (3.2.18) and (3.2.19) (hence (3.2.20)) into account we obtain (keeping in mind that $S_1 \neq 0$)

$$\frac{2}{S_1} \frac{S_1}{2} x A_1(y) = x A_1(y)$$

for all $x, y \in \mathbb{R}$; which actually means that taking an *arbitrary* additive function A_1 as f, we get that the pair (F, f) is a solution of (3.2.11) for $k = 1$. Of course, the solutions are mostly discontinuous.

Now, let us proceed to the case $k = 2$. Observe that f is now a diagonalization of a biadditive, symmetric function A_2. Similarly as in the previous cases, putting $x = y$ we obtain from (3.2.11)

$$S_2 F(x) = 2xf(x),$$

for every $x \in \mathbb{R}$, whence, in view of $S_2 \neq 0$,

$$F(x) = \frac{2}{S_2} x f(x) \tag{3.2.21}$$

for all $x \in \mathbb{R}$. Denote $\frac{2}{S_2}$ by C_2.

Let us substitute the formula (3.2.21) into (3.2.11). We obtain

$$C_2 \left[\sum_{i=1}^{n} \gamma_i \left(\alpha_i x + \beta_i y\right) f\left(\alpha_i x + \beta_i y\right) \right] = xf(y) + yf(x),$$

for all $x, y \in \mathbb{R}$. Using the biadditivity of f (and hence its rational homogeneity) we obtain hence

$$\begin{aligned}
&C_2 \sum_{i=1}^{n} \gamma_i \left(\alpha_i x + \beta_i y\right) \left(\alpha_i^2 A_2^*(x) + 2\alpha_i\beta_i A_2(x,y) + \beta_i^2 A_2^*(y)\right) \\
&\quad = C_2 \sum_{i=1}^{n} \gamma_i \left(\alpha_i^3 x A_2^*(x) + 2\alpha_i^2\beta_i x A_2(x,y) + \alpha_i\beta_i^2 x A_2^*(y)\right. \\
&\quad \left. + \alpha_i^2\beta_i y A_2^*(x) + 2\alpha_i\beta_i^2 y A_2(x,y) + \beta_i^3 y A_2^*(y)\right) \\
&\quad = C_2 \sum_{i=1}^{n} \gamma_i \left(\alpha_i^3 x A_2^*(x) + \beta_i^3 y A_2^*(y)\right) \\
&\quad + C_2 \sum_{i=1}^{n} \gamma_i \alpha_i^2 \beta_i \left(2x A_2(x,y) + y A_2^*(x)\right) \\
&\quad + C_2 \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i^2 \left(x A_2^*(y) + 2y A_2(x,y)\right) \\
&\quad = x A_2^*(y) + y A_2^*(x),
\end{aligned} \tag{3.2.22}$$

for all $x, y \in \mathbb{R}$. Now, comparing the terms of the same degree on both sides of (3.2.22) we get first that either

$$\sum_{i=1}^{n} \gamma_i \alpha_i^3 = \sum_{i=1}^{n} \gamma_i \beta_i^3 = 0 \tag{3.2.23}$$

or $A_2 = 0$. In the sequel we assume that $A_2 \neq 0$, hence (3.2.23) holds. In other words $S_2 = 3 \sum_{i=1}^{n} \gamma_i \left(\alpha_i^2 \beta_i + \alpha_i \beta_i^2\right)$. Let us compare the remaining terms. We get

$$C_2 \sum_{i=1}^{n} \gamma_i \alpha_i^2 \beta_i \left(2x A_2(x,y) + y A_2^*(x)\right) = y A_2^*(x),$$

and

$$C_2 \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i^2 \left(2y A_2(x,y) + x A_2^*(y)\right) = x A_2^*(y),$$

for all $x, y \in \mathbb{R}$. Putting $x = y$ above; and taking into account that $A_2 \neq 0$ we infer that $\sum_{i=1}^{n} \gamma_i \alpha_i^2 \beta_i = \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i^2 = \frac{1}{3C_2} = \frac{S_2}{6}$. Hence we may write

$$x A_2(x,y) = y A_2^*(x) \tag{3.2.24}$$

and

$$yA_2(x,y) = xA_2^*(y) \tag{3.2.25}$$

for all x, $y \in \mathbb{R}$. Putting $y = 1$ into (3.2.24) and (3.2.25) we obtain

$$A_2^*(x) = x^2 A_2^*(1) \tag{3.2.26}$$

for every $x \in \mathbb{R}$, hence f and F are continuous.

Now, let us pass to the situation where $k \geqslant 3$. In general, if $k \geqslant 3$ and f and F satisfy (3.2.11) then

$$f(x) = A_k^*(x),$$

for every $x \in \mathbb{R}$ and hence

$$F(x) = \frac{2}{S_k} x A_k^*(x),$$

for every $x \in \mathbb{R}$. Put $C_k := \frac{2}{S_k}$. We can write

$$\begin{aligned}
&C_k \sum_{i=1}^{n} \gamma_i \left[\alpha_i x \left(\sum_{j=0}^{k} \binom{k}{j} \alpha_i^j \beta_i^{k-j} A_k(x^j, y^{k-j}) \right) \right. \\
&\quad \left. + \beta_i y \left(\sum_{j=0}^{k} \binom{k}{j} \alpha_i^j \beta_i^{k-j} A_k(x^j, y^{k-j}) \right) \right] \\
&\quad = C_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{j=0}^{k} \binom{k}{j} \alpha_i^{j+1} \beta_i^{k-j} x A_k(x^j, y^{k-j}) \right) \right. \\
&\quad \left. + \left(\sum_{j=0}^{k} \binom{k}{j} \alpha_i^j \beta_i^{k+1-j} y A_k(x^j, y^{k-j}) \right) \right] \\
&\quad = C_k \left[\left(\sum_{i=1}^{n} \gamma_i \alpha_i^{k+1} \right) x A_k^*(x) + \left(\sum_{i=1}^{n} \gamma_i \beta_i^{k+1} \right) y A_k^*(y) \right] \\
&\quad + C_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{j=0}^{k-1} \binom{k}{j} \alpha_i^{j+1} \beta_i^{k-j} x A_k(x^j, y^{k-j}) \right) \right. \\
&\quad \left. + \left(\sum_{j=1}^{k} \binom{k}{j} \alpha_i^j \beta_i^{k+1-j} y A_k(x^j, y^{k-j}) \right) \right] \\
&\quad = C_k \left[\left(\sum_{i=1}^{n} \gamma_i \alpha_i^{k+1} \right) x A_k^*(x) + \left(\sum_{i=1}^{n} \gamma_i \beta_i^{k+1} \right) y A_k^*(y) \right] \\
&\quad + C_k \sum_{i=1}^{n} \gamma_i \left[\alpha_i \beta_i^k \left(x A_k^*(y) + k y A_k(x, y^{k-1}) \right) + \alpha_i^k \beta_i \left(k x A_k(x^{k-1}, y) + y A_k^*(x) \right) \right] \\
&\quad + C_k \sum_{i=1}^{n} \gamma_i \left[\sum_{j=2}^{k-1} \alpha_i^j \beta_i^{k+1-j} \left(\binom{k}{j-1} x A_k(x^{j-1}, y^{k+1-j}) + \binom{k}{j} y A_k(x^j, y^{k-j}) \right) \right] \\
&\quad = x A_k^*(y) + y A_k^*(x),
\end{aligned} \tag{3.2.27}$$

for all x, $y \in \mathbb{R}$. Comparing terms of equal degrees we infer that either $A_k = 0$ or $\sum_{i=1}^{n} \gamma_i \alpha_i^{k+1} = \sum_{i=1}^{n} \gamma_i \beta_i^{k+1} = 0$ (cf. (3.2.14)). Assume from now on that we are interested in nontrivial solutions of (3.2.11). Continuing comparisons of the terms on both sides of (3.2.27), we get for every $j \in \{2, \ldots, k-1\}$

$$C_k \sum_{i=1}^{n} \gamma_i \alpha_i^j \beta_i^{k+1-j} = 0,$$

(cf. (3.2.13)) for otherwise (putting $x = y$) we would get

$$\binom{k+1}{j} x A_k^*(x) = 0,$$

which is impossible. Note that from the above (3.2.15) and (3.2.16) follows. Taking this into account, as well as the definition of C_k and comparing the remaining terms in (3.2.27), we get

$$C_k \sum_{i=1}^{n} \gamma_i \alpha_i \beta_i^k \left(x A_k^*(y) + k y A_k(x, y^{k-1}) \right) = x A_k^*(y),$$

for all $x, y \in \mathbb{R}$. Using (3.2.15), we get hence

$$y A_k(x, y^{k-1}) = x A_k^*(y), \tag{3.2.28}$$

and analogously we infer

$$x A_k(x^{k-1}, y) = y A_k^*(x), \tag{3.2.29}$$

for all $x, y \in \mathbb{R}$. Let us put $x + y$ instead of x in (3.2.29). We obtain, after some easy though tedious calculations that the lefthand side is equal to

$$L := x \left[\sum_{j=0}^{k-1} \binom{k-1}{j} A_k(x^{k-1-j}, y^{j+1}) \right] + y \left[\sum_{j=0}^{k-1} \binom{k-1}{j} A_k(x^{k-1-j}, y^{j+1}) \right],$$

while the right-hand side is equal to

$$R := y \sum_{j=0}^{k} \binom{k}{j} A_k(x^{k-j}, y^j).$$

Comparing on both sides the terms of equal degree we obtain in particular the following sequence of equalities.

$$x A_k(x^{k-j-1}, y^{j+1}) = y A_k(x^{k-j}, y^j), \tag{3.2.30}$$

for $j \in \{0, \ldots, k-1\}$ and all $x, y \in \mathbb{R}$. Now, using (3.2.30) for $j \in \{0, \ldots, k-1\}$ we arrive at

$$y^k A_k^*(x) = y^{k-1} \left[y A_k^*(x) \right] = y^{k-1} \left[x A_k(x^{k-1}, y) \right] = \cdots = x^k A_k^*(y)$$

for every $x, y \in \mathbb{R}$, in other words, putting $y = 1$ we obtain

$$A_k^*(x) = A_k^*(1) x^k, \tag{3.2.31}$$

for every $x \in \mathbb{R}$, which means that A_k is continuous for $k \geqslant 3$ and thus the proof is finished. □

Remark 3.2.3. *(cf. Remark 3.3 in [25]) Using Lemma 3.1.1 exactly in the same way as we did in the proof of Theorem 3.2.1, we infer rather easily that if the functions F and f satisfy (3.2.11) then f must be a polynomial function. In the following simple example we observe that the function F is not necessarily polynomial.*

Example 3.2.1. *(cf. Example 1 in [25]) Observe that the equation*

$$F(x) - F(-x) = x f(y) + y f(x) \tag{3.2.32}$$

is satisfied by any even function F and $f = 0$.

The reason why the above example works is that the equation

$$F(x) - F(-x) = 0,$$

for all $x \in \mathbb{R}$, has solutions which are not polynomial. If we consider a general linear equation

$$\sum_{i=1}^{n} \gamma_i F(\alpha_i x + \beta_i y) = 0, \tag{3.2.33}$$

for all $x, y \in \mathbb{R}$, and we assume that at least one of the pairs (α_i, β_i) is linearly independent from all others then, using Theorem 2.2.1, it may be shown that every solution of (3.2.33) is a polynomial function. Therefore it is natural to formulate the following problem.

Problem 3.2.1. *(cf. Problem 1 in [25]) Let $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}, \gamma_i \neq 0, i = 1, \ldots, n$ be such that there exists an $i_0 \in \{1, \ldots, n\}$ satisfying*

$$\begin{vmatrix} \alpha_{i_0} & \beta_{i_0} \\ \alpha_i & \beta_i \end{vmatrix} \neq 0, \ i \neq i_0.$$

Is it possible that the functional equation (3.2.11) is satisfied by some functions f, F where F is not a polynomial function?

As we have seen (cf. Example 3.2.1) it is possible that equation (3.2.11) is satisfied by a pair (F, f) where F is not a polynomial function. However we will give some examples of particular forms of this equation which have only polynomial solutions and therefore we can apply Theorem 3.2.2 to solve these equations.

Proposition 3.2.2. *(cf. Proposition 3.4 in [25]) Let $\alpha_i, \beta_i, \gamma_i$, $i \in \{1, \ldots, n\}$ be real numbers such that*

$$\sum_{i=1}^{n} \gamma_i \neq 0 \tag{3.2.34}$$

holds and $\alpha_i + \beta_i = 1$, $i \in \{1, \ldots, n\}$. If the pair (F, f) of functions mapping $\mathbb{R}$ to $\mathbb{R}$ satisfies equation (3.2.11) then the functions F and f are polynomial.

Proof. Similarly as before, from Lemma 3.1.1 we know that f is a polynomial function. Now it is enough to take $x = y$ in (3.2.11) to show that also F must be polynomial. □

3.2.3 Applications

Now we show some examples of equations (with nontrivial solutions) which may be solved with the use of Proposition 3.2.2 and Theorem 3.2.2 .

Example 3.2.2. *(cf. Example 2 in [25]) Assume that functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$F(x) - 4F\left(\tfrac{x+y}{2}\right) + F(y) = xf(y) + yf(x), \tag{3.2.35}$$

for all x, $y \in \mathbb{R}$. Rearranging (3.2.35) in the form

$$yf(x) - F(x) = -f(y)x - 4F\left(\tfrac{x+y}{2}\right) + F(y),$$

for all x, $y \in \mathbb{R}$, we can see that f is a polynomial function of order at most 2. From Proposition 3.2.2 we know that also F is a polynomial function. Now we check the conditions of Theorem 3.2.2. If $k = 0$ then $f(x) = b$ for some constant $b \in \mathbb{R}$ and all $x \in \mathbb{R}$, further $S_0 = -2 \neq 0$ and, consequently, $F(x) = -bx$, for all $x \in \mathbb{R}$. Now let $k = 1$, then $S_1 = -2$,

$$\sum_{i=1}^{3} \gamma_i \alpha_i^2 = \sum_{i=1}^{3} \gamma_i \beta_i^2 = 0,$$

and again from Theorem 3.2.2 we infer that f is any additive function and $F(x) = -xf(x)$ for all $x \in \mathbb{R}$. If $k = 2$ then it is easy to see that the solutions of (3.2.35) must vanish. Thus the general solution of this equation is given by $f(x) = a(x) + b$ and $F(x) = -xa(x) - bx$, $x \in \mathbb{R}$, where $a : \mathbb{R} \to \mathbb{R}$ is additive and b is a constant.

Example 3.2.3. *(cf. Example 3 in [25]) Assume that functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$F(x) - 8F\left(\tfrac{x+y}{2}\right) + F(y) = xf(y) + yf(x), \tag{3.2.36}$$

for all $x,\, y \in \mathbb{R}$. Rearranging (3.2.36) in the form

$$yf(x) - F(x) = -f(y)x - 8F\left(\frac{x+y}{2}\right) + F(y),$$

for all $x,\, y \in \mathbb{R}$, we can see that f is a polynomial function of order at most 2. From Proposition 3.2.2 we know that also F is a polynomial function. Now we check the conditions of Theorem 3.2.2. If $k = 0$ then $f(x) = b$ for some constant $b \in \mathbb{R}$, and all $x \in \mathbb{R}$, further $S_0 = -6 \neq 0$ and, consequently, $F(x) = -\frac{b}{3}x$ for all $x \in \mathbb{R}$. Now let $k = 1$, then $S_1 = -6$ but this time

$$\sum_{i=1}^{3} \gamma_i \alpha_i^2 = \sum_{i=1}^{3} \gamma_i \beta_i^2 = -1 \neq 0$$

and again from Theorem 3.2.2 we infer that $f = F = 0$. If $k = 2$ then the solutions must be continuous since $S_2 = -6 \neq 0$, moreover

$$\sum_{i=1}^{3} \gamma_i \alpha_i^3 = \sum_{i=1}^{3} \gamma_i \beta_i^3 = 0$$

which means that $f(x) = cx^2$ and $F(x) = -\frac{c}{3}x^3$, $x \in \mathbb{R}$ satisfy (3.2.36). Thus the general solution of this equation is given by $f(x) = cx^2 + b$ and $F(x) = -\frac{c}{3}x^3 - \frac{b}{3}x$, $x \in \mathbb{R}$ where b, c are real constants.

3.2.4 A generalization of the Fechner-Gselmann equation

Observe that in equation (1.0.1) the left-hand side is the difference connected with the Cauchy equation. Since additive functions are monomial functions of order one, it is natural to ask whether this difference may be replaced by the difference connected with monomial function of higher orders or with the polynomial functions. In the next part of this chapter we consider functional equation constructed in such way.

Lemma 3.2.1. *(cf. Lemma 3.1 in [25]) Let n be a given positive integer, if the pair (F, f) of functions mapping $\mathbb{R}$ to $\mathbb{R}$ satisfies the equation*

$$\Delta_y^n F(x) = xf(y) + yf(x), \tag{3.2.37}$$

for all $x,\, y \in \mathbb{R}$, then f is a polynomial function of order at most $n+1$ and F is a polynomial function of order not greater than $n+2$.

Proof. We write (3.2.37) in the form

$$f(x)y - (-1)^n F(x) = -f(y)x - \sum_{i=1}^{n} (-1)^i \binom{n}{i} F(x + iy),$$

for all $x,\, y \in \mathbb{R}$. Similarly as before, using Lemma 3.1.1, we can see that f is a polynomial function of order at most $(n+1) + 1 - 1 = n + 1$. Indeed, observe that in the present situation we have $K_0 = \{(\mathrm{id}, i\mathrm{id}) : i \in \{1, \ldots, n\}\}$ and $K_1 = \{(0, \mathrm{id})\}$. Hence $\mathrm{card}(K_0 \cup K_1) = n + 1$ and $\mathrm{card}K_1 = 1$, whence the estimation follows (cf. (3.1.2)).

Further, applying the difference operator with the span y $(n + 2)-$times to the both sides of (3.2.37) we get

$$\Delta_y^{2n+2} F(x) = 0,$$

for all $x \in \mathbb{R}$ i.e. F is a polynomial function of order $2n + 1$.

Now consider any $k > n + 1$, the function f is a polynomial function of order smaller than k thus the monomial summand of F of order $k + 1$ satisfies (3.2.37) with $f = 0$. However the $n-$th difference does not vanish for monomial functions of order k. This means that summands of F of orders greater than $n + 2$ must be zero, i.e. F is a polynomial function of order at most $n + 2$. □

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Now we turn our attention to the equation where the left-hand side is the difference connected with the equation of monomial functions.

Lemma 3.2.2. *(cf. Lemma 3.2 in [25]) Let n be a given positive integer, if the pair (F, f) of functions mapping $\mathbb{R}$ to $\mathbb{R}$ satisfies the equation*

$$\Delta_y^n F(x) - n!F(y) = xf(y) + yf(x), \tag{3.2.38}$$

for all x, $y \in \mathbb{R}$, then f is a polynomial function of order at most $n+1$ and F is a polynomial function of order not greater than $n+2$.

Proof. We write (3.2.38) in the form

$$f(x)y - (-1)^n F(x) = -f(y)x - \sum_{i=1}^{n}(-1)^i \binom{n}{i} F(x+iy) - n!F(y),$$

for all x, $y \in \mathbb{R}$. We see that $K_0 = \{(\mathrm{id}, i\mathrm{id}) : i \in \{1, \ldots, n\}\} \cup \{(0, \mathrm{id})\}$ and $K_1 = \{(0, \mathrm{id})\}$. Hence $\mathrm{card}(K_0 \cup K_1) = n+1$ and $\mathrm{card}K_1 = 1$. Now applying again Lemma 3.1.1, we can see (cf. (3.1.2)) that f is a polynomial function of order at most $(n+1)+1-1 = n+1$. Further, applying the difference operator with the span y $(n+2)-$times to the both sides of (3.2.38) we get

$$\Delta_y^{2n+2} F(x) = 0,$$

for all $x \in \mathbb{R}$, i.e. F is a polynomial function of order $2n+1$.

Now, similarly as in the respective part of the proof of Lemma 3.2.1, we can see that the order of F cannot be greater than $n+2$. Indeed, summands of F of orders $k > n+2$ must satisfy (3.2.38) with the right-hand side equal to zero (since f has no terms of order $k-1$) which is impossible since the equation

$$\Delta_y^n F(x) - n!F(y) = 0,$$

for all x, $y \in \mathbb{R}$, characterizes monomial functions of order $n < k$. □

Now we can present the general solutions of equations (3.2.37) and (3.2.38).

Theorem 3.2.3. *(cf. Theorem 3.5 in [25]) A pair (F, f) of functions mapping $\mathbb{R}$ to $\mathbb{R}$ satisfies equation (3.2.37) if and only if F is a polynomial function of order at most $n-1$ and $f = 0$.*

Proof. From Lemma 3.2.1 we know that both f and F are polynomial functions. Take first $k \in \{0, 1, \ldots, n-2\}$, and assume that f is a monomial function of order k and that F is a monomial function of order $k+1$. We can see that $S_k = 0$ i.e. from Theorem 3.2.2 we obtain $f = 0$.

Now, take $k \in \{n-1, n, n+1\}$, then $S_k \neq 0$ and, as previously, assume that f is a monomial function of order k and that F is a monomial function of order $k+1$. We want to show that $f = 0$. Thus for the indirect proof assume that $f \neq 0$, then F is also nonzero. Observe that it leads to a contradiction. Indeed, equation (3.2.37) cannot be satisfied, since in the expression $\Delta_y^n F(x)$ we have the term of order $k+1$ with respect to y which is missing at the right-hand side.

We proved that $f = 0$, thus F obviously satisfies

$$\Delta_y^n F(x) = 0$$

for all x, $y \in \mathbb{R}$, i.e. F is a polynomial function of order at most $n-1$. □

In the next theorem we obtain the solution of equation (3.2.38).

Theorem 3.2.4. *(cf. Theorem 3.6 in [25]) Let* (F, f) *be a pair of functions mapping* $\mathbb{R}$ *to* $\mathbb{R}$. *If* $n = 1$ *then the solutions of* (3.2.38) *are of the form obtained in Proposition 3.2.1. If* $n > 2$ *then* F *is a monomial function of order* n *and* $f = 0$.

Proof. If $n = 1$ then (3.2.38) reduces to (1.0.1) which is already solved. Thus we may assume that $n \geqslant 2$. Using Lemma 3.2.2, we can see that function F and f are polynomial and as usually we will work with monomial functions. Thus let f and F be monomial functions of orders $k, k+1$; respectively. We want to show that $f = 0$. However, if $f \neq 0$ then the right-hand side which is of the form $xf(y)+yf(x)$, contains the term $yf(x)$ of order k with respect to the variable x. Such term is missing in the expression $\Delta_y^n F(x) - n!F(y)$, since $n \geqslant 2$. Therefore also in this case we have $f = 0$.

Using the equality $f = 0$ in (3.2.38), we get

$$\Delta_y^n F(x) - n!F(y) = 0,$$

for all $x, y \in \mathbb{R}$, for each monomial summand of F. This means that F is a monomial function of order n. □

Remark 3.2.4. *(cf. Remark 3.4 in [25]) It is interesting that we have a nice set of solutions only for the difference stemming from Cauchy's equation. Thus the case* $n = 1$ *in* (3.2.38) *is exceptional. It seems that the right-hand side of* (1.0.1) *must be suitably modified to get a similar effect for* $n > 1$.

We can add one more class of functional equations which may be solved with use of Theorem 3.2.2.

Proposition 3.2.3. *(cf. Proposition 3.7 in [25]) Let* β_i, $i \in \{1, \ldots, n\}$, γ_i, $i \in \{1, \ldots, n+1\}$ *be real numbers such that* (3.2.34) *holds. If the pair* (F, f) *of functions mapping* $\mathbb{R}$ *to* $\mathbb{R}$ *satisfies the equation*

$$\sum_{i=1}^{n} \gamma_i F(x + \beta_i y) + \gamma_{n+1} F(y) = xf(y) + yf(x), \tag{3.2.39}$$

for all $x, y \in \mathbb{R}$, *then functions* f *and* F *are polynomial.*

Proof. Similarly as before, from Lemma 3.1.1 we know that f is a polynomial function. Now it is enough to take $y = 0$ in (3.2.39) to show that also F must be polynomial. □

Remark 3.2.5. *(cf. Remark 3.5 in [25]) Note that equation* (3.2.39) *is a generalization of equations* (3.2.38) *and* (3.2.37). *However the methods used in Lemmas 3.2.1 and 3.2.2 were needed to show that* F *is polynomial because, in case of these equation, the condition* (3.2.34) *is not satisfied.*

We end the chapter with a remark connecting the results obtained here with the topic called alienation of functional equations (for some details concerning the problem of alienation of functional equations see the survey paper of R. Ger and M. Sablik[15]).

Remark 3.2.6. *(cf. Remark 3.6 in [25]) Consider two equations:*

$$xf(y) + yf(x) = 0 \tag{3.2.40}$$

which is satisfied only by $f = 0$ *and*

$$\sum_{i=1}^{n} \gamma_i F(\alpha_i x + \beta_i y) = 0 \tag{3.2.41}$$

which usually has some solutions (depending on n *and constants involved). Results concerning equation* (3.2.11) *may be viewed from the perspective of the so called alienation of functional* *equations. Any pair of the form* $(F, 0)$ *where* F *satisfies* (3.2.41) *is clearly a solution of* (3.2.11). *Interesting is the question if* (3.2.11) *may have solutions of a different nature. As we proved in case of some equations there are only solutions of this kind whereas in some other cases new solutions appear. Thus, in fact, we have examples of alienation and nonalienation of equations of this kind. It may even happen that for monomial functions of some order, some equations are alien and for other orders the same equations are not alien. This effect is similar to the approach presented in [36] by T. Szostok.*

Chapter 4

Functional Equation Characterizing Polynomial Functions and an Algorithm

4.1 Generalized right-hand side of the Fechner-Gselmann equation

We continue the investigation presented in Chapter 3 (see [25]), where we generalized the left-hand side of the Fechner-Gselmann equation, namely (1.0.2). This chapter presents our results in [28], where we solve the generalized right-hand side of the Fechner and Gselmann equation, that is we find general solutions of the following functional equation.

$$F(x+y) - F(x) - F(y) = \sum_{i=1}^{m} (a_i x + b_i y) f(\alpha_i x + \beta_i y) \tag{4.1.1}$$

for all $x, y, a_i, b_i \in \mathbb{R}$ and $\alpha_i, \beta_i \in \mathbb{Q}$. It turns out that under some mild assumption, the pair (F, f) solving (4.1.1) happens to be a pair of polynomial functions and, in some important cases, just the usual polynomials (even though we assume no regularity of solutions a priori). In the second part of this chapter, we formulate an algorithm written in the computer algebra system Maple which determines the polynomial solutions of the functional equations belonging to the class (4.1.1).

While proving the main result in Chapter 3 (cf. Theorem 3.2.2 in Chapter 3 and Theorem 3.3 in [25]), we observed that the behaviour of solutions depends on the sequence $(S_k)_{k\in\mathbb{N}\cup\{0\}}$ given by

$$S_k = \sum_{i=1}^{n} \gamma_i (\alpha_i + \beta_i)^{k+1},$$

for all $k \in \mathbb{N}\cup\{0\}$. A characteristic feature of equation (4.1.1) is the dependence of the existence of solutions on the sequences $(R_k)_{k\in\mathbb{N}\cup\{0\}}$ given by

$$R_k = \sum_{i=1}^{m} (a_i + b_i)(\alpha_i + \beta_i)^k \tag{4.1.2}$$

for all $k \in \mathbb{N} \cup \{0\}$.

4.1.1 Theoretical Results

Let us start with the result showing that in some cases any solution (F, f) of (4.1.1) consists of polynomial functions.

Theorem 4.1.1. *(cf. Theorem 2.1 in [28]) Let $a_i, b_i, \alpha_i, \beta_i \in \mathbb{R}$, $i \in \{1, \cdots, m\}$. Suppose further that the pair of functions (F, f) mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfies equation (4.1.1). If there exists an $i \in \{1, \cdots, m\}$ such that*

$$\begin{vmatrix} \alpha_i & \beta_i \\ a_i & b_i \end{vmatrix} \neq 0, \tag{4.1.3}$$

then f is a polynomial function of degree not greater than $2m$ and F is a polynomial function of degree not greater than $2m+1$.

Proof. By change of variable and applying Lemma 3.1.1, we obtain that f is a polynomial function of degree not greater than $2m$. Now, it is enough to take $y = h$ in (4.1.1) and applying $2m+2$ times the Fréchet operator to both sides of (4.1.1) we see that the right-hand side vanishes and so does the left-hand side. This means that F is a polynomial function of order not greater than $2m+1$. □

Now, we proceed to the next theorem. However, it is now enough to assume that the pair of functions (F, f) satisfying equation (4.1.1) are monomials (see [25], Remark 3.2).

Theorem 4.1.2. *(cf. Theorem 2.2 in [28]) Let $a_i, b_i \in \mathbb{R}, \alpha_i, \beta_i \in \mathbb{Q}, i \in \{1, \ldots, m\}$. Further, let $(R_k)_{k \in \mathbb{N} \cup \{0\}}$ be defined by (4.1.2). Assume that $R_k \neq 0$ for some $k \in \mathbb{N} \cup \{0\}$, and that equation (4.1.1) is satisfied by the pair $(F, f) : \mathbb{R} \longrightarrow \mathbb{R}$, of monomial functions of order $k+1$ and k, respectively. The following assertions hold*

(i) if $k = 0$ then $f = 0$ and F is arbitrary additive function.

(ii) if $k \neq 0$ then either $f = F = 0$ is the only solution of (4.1.1), or $\sum_{i=1}^{m} a_i \alpha_i^k = \sum_{i=1}^{m} b_i \beta_i^k = 0$ while F is given by $F(x) = \frac{R_k}{2^{k+1}-2} x f(x)$. Moreover, for non-trivial f we see that either

(a) $\sum_{i=1}^{m} b_i \alpha_i^{j+1} \beta_i^{k-j-1} = \sum_{i=1}^{m} a_i \alpha_i^j \beta_i^{k-j}$ for each $j \in \{0, \cdots, k-1\}$ and f is an arbitrary k-monomial function, or

(b) $\sum_{i=1}^{m} b_i \alpha_i^{j+1} \beta_i^{k-j-1} \neq \sum_{i=1}^{m} a_i \alpha_i^j \beta_i^{k-j}$ for each $j \in \{0, \cdots, k-1\}$ and f is necessarily a continuous monomial function of order k and so is F of order $k+1$.

Proof. Suppose that $k = 0$. Then $f = \text{const} = A_0$ and F is additive. Putting $x = y$ in (4.1.1) we obtain

$$0 = \sum_{i=1}^{m} (a_i + b_i) x A_0$$

i.e.

$$0 = R_0 x A_0$$

for every $x \in \mathbb{R}$. Since $R_0 \neq 0$, it follows that $A_0 = 0 = f$.

Suppose that $k = 1$, we obtain that $f = A_1$ is additive and $F = B_2^*$ is a quadratic function, or, in other words, diagonalization of a biadditive function. Putting $x = y$ in (4.1.1) we obtain (taking into account the rational homogeneity of f)

$$2B_2^*(x) = \left(\sum_{i=1}^{m} (a_i + b_i)(\alpha_i + \beta_i) \right) x A_1(x),$$

whence

$$2F(x) = R_1 x A_1(x)$$

for every $x \in \mathbb{R}$, denoting $D_1 := \frac{R_1}{2}$ we get

$$F(x) = D_1 x A_1(x)$$

for every $x \in \mathbb{R}$. Substituting the above into (4.1.1) we obtain (taking into account the rational homogeneity of A_1)

$$D_1\left[(x+y)A_1(x+y) - xA_1(x) - yA_1(y)\right] = \sum_{i=1}^{m}(a_i x + b_i y)(\alpha_i A_1(x) + \beta_i A_1(y))$$

and further

$$\begin{aligned} D_1\left[xA_1(y) + yA_1(x)\right] = &\left(\sum_{i=1}^{m} a_i\alpha_i\right) xA_1(x) + \left(\sum_{i=1}^{m} b_i\alpha_i\right) yA_1(x) \\ &+ \left(\sum_{i=1}^{m} a_i\beta_i\right) xA_1(y) + \left(\sum_{i=1}^{m} b_i\beta_i\right) yA_1(y) \end{aligned} \tag{4.1.4}$$

for all $x, y \in \mathbb{R}$. Comparing terms of the same degree on both sides of the above equation, we obtain

$$\left(\sum_{i=1}^{m} a_i\alpha_i\right) xA_1(x) = 0$$

for all $x \in \mathbb{R}$, and symmetrically

$$\left(\sum_{i=1}^{m} b_i\beta_i\right) yA_1(y) = 0$$

for all $y \in \mathbb{R}$. Both of these equations hold if either $A_1 = 0$ or

$$\sum_{i=1}^{m} a_i\alpha_i = \sum_{i=1}^{m} b_i\beta_i = 0. \tag{4.1.5}$$

Now if $A_1 = 0$ then also $F = 0$. Let us look for non-zero solutions of (4.1.1). The existence of a nontrivial A_1 implies that (4.1.5) holds. So, in this case we have

$$D_1 = \frac{R_1}{2} = \frac{1}{2}\sum_{i=1}^{m} a_i\beta_i + \frac{1}{2}\sum_{i=1}^{m} b_i\alpha_i. \tag{4.1.6}$$

Taking into account (4.1.4), (4.1.5) and (4.1.6) we obtain

$$\left(\sum_{i=1}^{m} b_i\alpha_i - \sum_{i=1}^{m} a_i\beta_i\right)(xA_1(y) - yA_1(x)) = 0 \tag{4.1.7}$$

for all $x, y \in \mathbb{R}$. From (4.1.7) we see that either

$$\sum_{i=1}^{m} b_i\alpha_i = \sum_{i=1}^{m} a_i\beta_i$$

which leads to a situation where A_1 can be an arbitrary (in particular discontinuous) additive function and we get that the pair (F, f) is a solution of (4.1.1), or

$$yA_1(x) = xA_1(y),$$

for all $x, y \in \mathbb{R}$. Putting $y = 1$ in the above equation we have

$$A_1(x) = xA_1(1),$$

for every $x \in \mathbb{R}$, hence f and F are continuous.

Now, let us proceed to the case $k = 2$. Assume that $f = A_2^*$ is a diagonalization of a biadditive symmetric function and $F = B_3^*$ is a diagonalization of a triadditive symmetric function. Putting $x = y$ in (4.1.1) we obtain (taking into account the rational homogeneity of f)

$$6B_3^*(x) = \left(\sum_{i=1}^{m}(a_i + b_i)(\alpha_i + \beta_i)^2\right) xA_2^*(x)$$

or

$$6F(x) = R_2 xA_2^*(x)$$

for every $x \in \mathbb{R}$. Denoting $D_2 := \frac{R_2}{6}$ we get

$$F(x) = D_2 xA_2^*(x)$$

for every $x \in \mathbb{R}$. Substituting the above into (4.1.1) we obtain (taking into account the rational homogeneity of A_2^*)

$$\begin{aligned} &D_2\left[(x+y)A_2^*(x+y) - xA_2^*(x) - yA_2^*(y)\right] \\ &\quad = \sum_{i=1}^{m}(a_i x + b_i y)(\alpha_i^2 A_2^*(x) + 2\alpha_i\beta_i A_2(x,y) + \beta_i^2 A_2^*(y)) \end{aligned}$$

whence

$$\begin{aligned} &D_2\left[2xA(x,y) + xA_2^*(y) + 2yA_2(x,y) + yA_2^*(x)\right] \\ &\quad = \left(\sum_{i=1}^{m} a_i\alpha_i^2 x + b_i\alpha_i^2 y\right) A_2^*(x) + 2\left(\sum_{i=1}^{m} a_i\alpha_i\beta_i x + b_i\alpha_i\beta_i y\right) A_2(x,y) \\ &\quad + \left(\sum_{i=1}^{m} a_i\beta_i^2 x + b_i\beta_i^2 y\right) A_2^*(y) \end{aligned} \tag{4.1.8}$$

for all $x, y \in \mathbb{R}$. Comparing terms of the same degree on both sides of the above equation, we obtain

$$\left(\sum_{i=1}^{m} a_i\alpha_i^2\right) xA_2^*(x) = 0,$$

for all $x \in \mathbb{R}$, and symmetrically

$$\left(\sum_{i=1}^{m} b_i\beta_i^2\right) yA_2^*(y) = 0,$$

for all $y \in \mathbb{R}$. Both of these equations hold if either $A_2 = 0$ or

$$\sum_{i=1}^{m} a_i\alpha_i^2 = \sum_{i=1}^{m} b_i\beta_i^2 = 0. \tag{4.1.9}$$

In the sequel we assume $A_2 \neq 0$, hence (4.1.9) holds. So, in this case we have

$$D_2 = \frac{R_2}{6} = \frac{1}{6}\sum_{i=1}^{m} a_i\beta_i^2 + \frac{1}{6}\sum_{i=1}^{m} b_i\alpha_i^2 + \frac{1}{3}\sum_{i=1}^{m} a_i\beta_i + \frac{1}{3}\sum_{i=1}^{m} b_i\alpha_i\beta_i. \tag{4.1.10}$$

Comparing the remaining terms of (4.1.8), we get

$$\left(2D_2 - 2\sum_{i=1}^{m} a_i\alpha_i\beta_i\right) xA_2(x,y) = \left(\sum_{i=1}^{m} b_i\alpha_i^2 - D_2\right) yA_2^*(x) \tag{4.1.11}$$

and

$$\left(2D_2 - 2\sum_{i=1}^{m} b_i\alpha_i\beta_i\right) yA_2(x,y) = \left(\sum_{i=1}^{m} a_i\beta_i^2 - D_2\right) xA_2^*(y) \tag{4.1.12}$$

for all $x, y \in \mathbb{R}$. Putting $x = y$ in (4.1.11) and (4.1.12) and taking into account that $A_2 \neq 0$ we infer that

$$D_2 = \tfrac{2}{3} \sum_{i=1}^{m} b_i \alpha_i \beta_i + \tfrac{1}{3} \sum_{i=1}^{m} a_i \beta_i^2 = \tfrac{2}{3} \sum_{i=1}^{m} a_i \alpha_i \beta_i + \tfrac{1}{3} \sum_{i=1}^{m} b_i \alpha_i^2.$$

Now substituting D_2 in (4.1.11) and (4.1.12) we obtain

$$\left(\sum_{i=1}^{m} b_i \alpha_i^2 - \sum_{i=1}^{m} a_i \alpha_i \beta_i \right) x A_2(x, y) = \left(\sum_{i=1}^{m} b_i \alpha_i^2 - \sum_{i=1}^{m} a_i \alpha_i \beta_i \right) y A_2^*(x) \tag{4.1.13}$$

and

$$\left(\sum_{i=1}^{m} a_i \beta_i^2 - \sum_{i=1}^{m} b_i \alpha_i \beta_i \right) y A_2(x, y) = \left(\sum_{i=1}^{m} a_i \beta_i^2 - \sum_{i=1}^{m} b_i \alpha_i \beta_i \right) x A_2^*(y) \tag{4.1.14}$$

for all $x, y \in \mathbb{R}$. Let us take into account (4.1.13). We see that either

a) $\sum_{i=1}^{m} b_i \alpha_i^2 = \sum_{i=1}^{m} a_i \alpha_i \beta_i$, or

b) $A_2 = 0$.

Suppose that case a) holds. Then A_2 is arbitrary. In particular, choosing x and y properly, we can assure that $x A_2(x, y) \neq y A_2^*(x)$ (of course A_2 has to be discontinuous then, because for a continuous A_2 we get $A_2(x, y) = cxy$ for some constant $c \in \mathbb{R}$ and all $x, y \in \mathbb{R}$). Fix the chosen x and y and let us pass to (4.1.14). If we interchange x and y, we easily see that (4.1.14) holds only if $\sum_{i=1}^{m} a_i \beta_i^2 = \sum_{i=1}^{m} b_i \alpha_i \beta_i$. Thus we see that either

$$\sum_{i=1}^{m} b_i \alpha_i^2 = \sum_{i=1}^{m} a_i \alpha_i \beta_i \text{ and } \sum_{i=1}^{m} a_i \beta_i^2 = \sum_{i=1}^{m} b_i \alpha_i \beta_i$$

(and A_2 is any biadditive function) or

$$x A_2(x, y) = y A_2^*(x) \tag{4.1.15}$$

and

$$y A_2(x, y) = x A_2^*(y) \tag{4.1.16}$$

for all $x, y \in \mathbb{R}$. Putting $y = 1$ into (4.1.15) and (4.1.16) we obtain

$$A_2^*(x) = x^2 A_2^*(1) \tag{4.1.17}$$

for every $x \in \mathbb{R}$, hence f and F are continuous. Let us observe that if case a) holds, and consequently, we also have

$$\sum_{i=1}^{m} a_i \beta_i^2 = \sum_{i=1}^{m} b_i \alpha_i \beta_i,$$

then any discontinuous A_2 (together with B_3 generated by $x A_2^*(x)$) satisfies (4.1.1).

Now, consider the general case $k \geqslant 3$ and $f = A_k^*$. Similarly as in the previous cases, putting $x = y$ we obtain from (4.1.1)

$$(2^{k+1} - 2) F(x) = R_k x A_k^*(x)$$

for all $x \in \mathbb{R}$. Denote $\frac{R_k}{2^{k+1}-2}$ by D_k we have $F(x) = D_k x A_k^*(x)$ for all $x \in \mathbb{R}$. Substituting into (4.1.1) we obtain

$$D_k \left[(x + y) A_k^*(x + y) - x A_k^*(x) - y A_k^*(y) \right] = \sum_{i=1}^{m} (a_i x + b_i y) A_k^*(\alpha_i x + \beta_i y)$$

or

$$D_k\left[(x+y)\left(\sum_{j=0}^{k}\binom{k}{j}A_k(x^j,y^{k-j})\right)-xA_k^*(x)-yA_k^*(y)\right]$$
$$=\sum_{i=1}^{m}(a_ix+b_iy)\left(\sum_{j=0}^{k}\binom{k}{j}\alpha_i^j\beta_i^{k-j}A_k(x^j,y^{k-j})\right)$$

whence

$$D_k\left[\left(\sum_{j=0}^{k}\binom{k}{j}xA_k(x^j,y^{k-j})\right)+\left(\sum_{j=0}^{k}\binom{k}{j}yA_k(x^j,y^{k-j})\right)\right]$$
$$-D_k\left[xA_k^*(x)-yA_k^*(y)\right]=\sum_{i=1}^{m}a_i\left(\sum_{j=0}^{k}\binom{k}{j}\alpha_i^j\beta_i^{k-j}xA_k(x^j,y^{k-j})\right)$$
$$+\sum_{i=1}^{m}b_i\left(\sum_{j=0}^{k}\binom{k}{j}\alpha_i^j\beta_i^{k-j}yA_k(x^j,y^{k-j})\right)$$

or

$$D_k\left[\left(\sum_{j=0}^{k-1}\binom{k}{j}xA_k(x^j,y^{k-j})\right)+\left(\sum_{j=1}^{k}\binom{k}{j}yA_k(x^j,y^{k-j})\right)\right]$$
$$=\left(\sum_{i=1}^{m}a_i\alpha_i^k\right)xA_k^*(x)+\sum_{i=1}^{m}a_i\left(\sum_{j=0}^{k-1}\binom{k}{j}\alpha_i^j\beta_i^{k-j}xA_k(x^j,y^{k-j})\right)$$
$$+\left(\sum_{i=1}^{m}b_i\beta_i^k\right)yA_k^*(y)+\sum_{i=1}^{m}b_i\left(\sum_{j=1}^{k}\binom{k}{j}\alpha_i^j\beta_i^{k-j}yA_k(x^j,y^{k-j})\right)\qquad(4.1.18)$$

for all $x,y\in\mathbb{R}$. Comparing terms of equal degrees we observe that either $A_k=0$ or $\sum_{i=1}^{m}a_i\alpha_i^k=\sum_{i=1}^{m}b_i\beta_i^k=0$. For the nontrivial solutions of (4.1.1) let $A_k\neq 0$ and rewriting (4.1.18) we get

$$D_k\left[\left(\sum_{j=0}^{k-1}\binom{k}{j}xA_k(x^j,y^{k-j})\right)+\left(\sum_{j=0}^{k-1}\binom{k}{j+1}yA_k(x^{j+1},y^{k-j-1})\right)\right]$$
$$=\sum_{i=1}^{m}a_i\left(\sum_{j=0}^{k-1}\binom{k}{j}\alpha_i^j\beta_i^{k-j}xA_k(x^j,y^{k-j})\right)$$
$$+\sum_{i=1}^{m}b_i\left(\sum_{j=0}^{k-1}\binom{k}{j+1}\alpha_i^{j+1}\beta_i^{k-j-1}yA_k(x^{j+1},y^{k-j-1})\right).\qquad(4.1.19)$$

Comparing on both sides of (4.1.19) the terms of equal degree we obtain the following sequence of equalities

$$\left(\binom{k}{j+1}D_k-\binom{k}{j+1}\sum_{i=1}^{m}b_i\alpha_i^{j+1}\beta_i^{k-j-1}\right)yA_k(x^{j+1},y^{k-j-1})$$
$$=\left(\binom{k}{j}\sum_{i=1}^{m}a_i\alpha_i^j\beta_i^{k-j}-\binom{k}{j}D_k\right)xA_k(x^j,y^{k-j})\qquad(4.1.20)$$

for $j\in\{0,\dots,k-1\}$ and for all $x,y\in\mathbb{R}$. Now, using (4.1.20) we arrive at

$$\prod_{j=0}^{k-1}\left(D_k-\sum_{i=1}^{m}b_i\alpha_i^{j+1}\beta_i^{k-j-1}\right)y^kA_k^*(x)=\prod_{j=0}^{k-1}\left(\sum_{i=1}^{m}a_i\alpha_i^j\beta_i^{k-j}-D_k\right)x^kA_k^*(y)\qquad(4.1.21)$$

for all $x,y\in\mathbb{R}$. From (4.1.21) we observe that either

$$D_k=\frac{R_k}{2^{k+1}-2}=\sum_{i=1}^{m}b_i\alpha_i^{j+1}\beta_i^{k-j-1}=\sum_{i=1}^{m}a_i\alpha_i^j\beta_i^{k-j}$$

for each $j \in \{0, \ldots, k-1\}$ which leads to a situation where A_k may be an arbitrary (in particular, discontinuous) k-additive function and we get that the pair (F, f) is a solution of (4.1.1), or

$$y^k A_k^*(x) = x^k A_k^*(y)$$

for all $x, y \in \mathbb{R}$, putting $y = 1$ we obtain

$$A_k^*(x) = x^k A_k^*(1)$$

for every $x \in \mathbb{R}$, which leads to the continuous solutions of (4.1.1) for $k \geqslant 3$. □

Remark 4.1.1. *(cf. Remark 2.1 in [28]) Let us observe that the assumption (4.1.3) is essential. Indeed, let us restrict to the case $m = 1$, and suppose that $a_i, b_i, \alpha_i, \beta_i$, $i \in \{1, \ldots, m\}$ be real numbers such that*

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ a_1 & b_1 \end{vmatrix} = 0.$$

Then vectors (α_1, β_1) and (a_1, b_1) are linearly dependent, i.e. either $\alpha_1 = \beta_1 = 0$ or there exists a $\lambda \in \mathbb{R}$ such that

$$(a_1, b_1) = \lambda(\alpha_1, \beta_1). \tag{4.1.22}$$

In the first case we can reformulate (4.1.1) as

$$F(x+y) - F(x) - F(y) = (a_1 x + b_1 y) f(0),$$

and thus if f is an arbitrary function satisfying $f(0) = 0$, we get additivity of F, while f needs not to be a polynomial function.

On the other hand, if $\lambda = 0$ then (4.1.1) takes the form

$$F(x+y) - F(x) - F(y) = (0x + 0y) f(\alpha_1 x + \beta_1 y) = 0,$$

and thus F is additive, and f may be arbitrary.

Finally, if $\lambda \neq 0$ then putting $z := \alpha_1 x + \beta_1 y$, we get

$$F(x+y) - F(x) - F(y) = \lambda z f(z),$$

whence (assuming that $\alpha_1 \neq 0$) we get

$$F\left(\frac{z}{\alpha_1} + \frac{(\alpha_1 - \beta_1)y}{\alpha_1}\right) - F\left(\frac{z}{\alpha_1} - \frac{\beta_1 y}{\alpha_1}\right) - F(y) = \lambda z f(z). \tag{4.1.23}$$

Putting in the above equation $y = 0$ we get

$$-F(0) = \lambda z f(z),$$

for every $z \in \mathbb{R}$. Moreover, putting $y = z = 0$ in (4.1.23) we get $F(0) = 0$, and hence

$$0 = \lambda z f(z)$$

for every $z \in \mathbb{R}$. It follows that

$$f(z) = \begin{cases} 0, & z \neq 0, \\ \textit{arbitrary}, & z = 0. \end{cases}$$

It is clear that at any case, f needs not to be polynomial.

Now, observe that it is possible that (4.1.3) holds and (4.1.1) is satisfied by a pair (F, f) of functions where f is not a polynomial function.

Example 4.1.1. *(cf. Example 1 in [28]) Observe that the equation*

$$F(x+y) - F(x) - F(y) = xf(x+y) + xf(-x-y) \tag{4.1.24}$$

for all $x, y \in \mathbb{R}$, is satisfied by an arbitrary additive function F and any odd function f.

Therefore, we will give particular forms of (4.1.1) which have only polynomial solutions.

Corollary 4.1.1. *(cf. Corollary 2.1 in [28]) Let $a_i, b_i, \alpha_i, \beta_i$, $i \in \{1, \ldots, m\}$ be real numbers such that (4.1.3) holds and*

$$a_i \alpha_i = 0 = b_i \beta_i \tag{4.1.25}$$

for each $i \in \{1, \cdots, m\}$. If the pair (F, f) of functions mapping $\mathbb{R}$ to $\mathbb{R}$ satisfies equation (4.1.1) then F and f are polynomials.

Proof. The proof is a direct application of Lemma 3.1.1 and Fréchet operator. □

Remark 4.1.2. *(cf. Remark 2.2 in [28]) If (4.1.25) holds then checking conditions of Theorem 4.1.2 for $k \geqslant 3$ is irrelevant because it leads to only trivial solutions ($A_k = f = 0$, and $F = 0$). Therefore, suppose if a pair of functions (F, f) satisfies (4.1.1), and (4.1.25) holds then f is a polynomial function of degree not greater than 2 and F is a polynomial function of degree not greater than 3. Note that in the case $m = 2$, $\alpha_2 = \beta_1 = a_1 = b_2 = 1$, $\alpha_1 = \beta_2 = a_2 = b_1 = 0$ we get the equation (1.0.1).*

Now, we develop an algorithm written in the computer algebra system Maple which takes into account the above results to determine the polynomial solutions of the functional equations belonging to the class (4.1.1). The main motivation for writing such an algorithm is that, solving even simple equations belonging to class (4.1.1) needs a long and tiresome calculation.

4.2 Description of the Algorithm

Taking into account Corollary 4.1.1, Lemma 3.1.1, Theorem 4.1.1, Remark 4.1.2, and Theorem 4.1.2 we formulate a procedure psfe(polynomial solution of functional equation) written in Maple program which can be used to obtain the polynomial solutions of functional equations of type (4.1.1).

The procedure is executed with the command

$$psfe(F(x+y) - F(x) - F(y) - \sum_{i=1}^{m} (a_i x + b_i y) f(\alpha_i x + \beta_i y),$$
$$[(a_1 x + b_1 y) f(\alpha_1 x + \beta_1 y), \cdots, (a_m x + b_m y) f(\alpha_m x + \beta_m y)])$$

where the first parameter of psfe is the functional equation to be solved of the form (4.1.1) and the second parameter is the list of the terms on the right-hand-side of the functional equation. If the parameters are not of this form then we get an error message.

In order for the algorithm to work properly, the unknown functions must be a function of F on the left-hand side and a function of f multiplied by a variable of x or y on the right-hand side otherwise we get an error message. If the difference between the number of terms in the first parameter(after conversion to a list by the algorithm) of psfe and the number of terms in the second parameter of psfe is not equal to 3 then we get an error message. If some terms appear in the second parameter of psfe and don't appear in the first parameter(after conversion to a list by the algorithm) of psfe then we get an error message. If the 3 terms on the left-hand side of (4.1.1) are not contained in the first parameter of psfe then we get an error message.

If Corollary 4.1.1 is not satisfied that is (4.1.3) and (4.1.25) is not valid for all terms in the second parameter of psfe then we get an error message. Let us consider the functional equation given in Example 1, it is easy to see that equation (4.1.24) belongs to class (4.1.1) but does not satisfy (4.1.25). In such a situation the algorithm outputs an error message. If we would like to solve equation (4.1.24) with our algorithm, we have to use the input

> psfe(F(x+y)-F(x)-F(y)-xf(x+y)-xf(-x-y), [xf(x+y), xf(-x-y)]);

We obtain the output,

> Error, (in psfe) No solution or wrong data input.

However, provided that the input is given in a correct form, we obtain the polynomial solutions of the functional equation given in the first parameter of psfe. The algorithm applies Lemma 3.1.1 and Theorem 4.1.1 to determine k(the potential degree of f). If $k \neq 0$ then the algorithm takes into account Remark 4.1.2 and applies conditions of Theorem 4.1.2 up to $k = 2$ to obtain the exact polynomial solutions.

4.3 Computer code

```
with(ArrayTools):
psfe:=proc(h,t)
local a,fd,fdd,ch,eb,ap,app,tt,tp,dsa,
L,R,k,ff,FF,p,i,nm,mn,j,dsp,dse,aa,
bb,hgg,hg,hgb,d,jj,ghh,gh,ee;
a:= convert(h,list): fd:=Array([]):
fdd:=Array([]): L:=0:
R:=0:  k:=0:  ff :=0:    FF:=0:
ch:=Array([]):   eb:=Array([]):
for i in [op(a), op(t)]do
    if nops(i) = 1 then
        Append(ch, op(0, i))
    else
        Append(ch, op(0, op(nops(i), i)))
    end if:
end do:
for i in convert(ch, set) do
     if has([F, xf, yf], i) = false then
        ERROR("No solution or wrong data input")
      end if:
end do:
if `subset`(convert([F(x+y), -F(x), -F(y)], set), convert(a, set)) = false then
    ERROR("No solution or wrong data input")
end if:
if `subset`(convert(-t, set), convert(a, set)) = false then
    print(convert(-t, set));
```

```
     print(convert(a, set));
     ERROR("No solution or wrong data input")
end if:
if nops(convert(a, set))-nops(convert(t, set)) <> 3 then
     ERROR("No solution or wrong data input")
end if:
for i in t do
    if nops(i) = 1 then
        ap := op(0, i);
        if has(ap, xf) or has(ap, yf) then
            Append(eb, coeff(ap, yf)*coeff(op(1, i), y));
            Append(eb, coeff(ap, xf)*coeff(op(1, i), x))
        end if:
     else
         app := op(0, op(2, i));
         if has(app, xf) or has(app, yf) then
             Append(eb, coeff(app, yf)*op(1, i)*coeff(op(1, op(nops(i), i)), y));
             Append(eb, coeff(app, xf)*op(1, i)*coeff(op(1, op(nops(i), i)), x))
          end if:
       end if:
 end do:
 if convert(eb, set) <> {0} and convert(eb, set) <> {} then
     ERROR("No solution or wrong data input")
 end if:
 for p in a do
     if has(p, [F(x), yf(x)]) then
         L := L-p
     else
         R := R+p
     end if:
 end do:
 fd := Array([]); fdd := Array([]);
 for i in convert(R, list) do
     if nops(i) = 1 then
         nm := op(nops(i), i)
     else
         nm := op(1, op(nops(i), i))
     end if:
     if  has(nm, x) or has(nm, y) then
         Append(fd, [coeff(nm, x)*identify, coeff(nm, y)*identify])
     end if:
 end do:
 for i in convert(R, list) do
     if nops(i) = 1 then
         nm := op(0, i)
     end if:
     if has(nm, yf) or has(nm, xf) then
         Append(fdd, [coeff(op(nops(i), i), x)*identify, coeff(op(nops(i), i), y))
     end if:
 end do:
```

```
for i in convert(R, list) do
    if nops(i) = 2 then
        mn := op(0, op(nops(i), i))
    end if:
    if has(mn, yf) or has(mn, xf) then
        Append(fdd, [coeff(op(1, op(nops(i), i)), x)*identify, coeff(op(1, op(i)))
     end if:
end do:
if nops(L) = 1 then
     k := k+numelems(convert(fd, set))+numelems(convert(fdd, set))-2
else
     k := k+numelems(convert(fd, set))+numelems(convert(fdd, set))-1
end if:
for j from 0 by 1 to k do
     if j =0 then
          ff:=ff+0;
          FF:=FF + A[1](x);
     end if:
end do:
dsp := Array([]); dse := Array([]);
for i in t do
    if nops(i) = 1 then
        aa := op(0, i);
        if has(aa, xf) or has(aa, yf)) then
            tt := Array([coeff(aa, xf), coeff(aa, yf)]);
            tp := Array([coeff(op(1, i), x), coeff(op(1, i), y)]);
            Append(dsp, tt(1)+tt(2));
            Append(dse, tp(1)+tp(2))
         end if:
    else
        bb := op(0, op(2, i));
        if has(bb, xf) or has(bb, yf) then
            tt := Array([coeff(bb, xf)*op(1, i), coeff(bb, yf)*op(1, i)]);
            tp := Array([coeff(op(1, op(nops(i), i)), x), coeff(op(1,op(nops(i)]);
             Append(dsp, tt(1)+tt(2));
             Append(dse, tp(1)+tp(2))
         end if:
    end if:
 end do:
 dsa := nops(t);
 if k = 0 then
     k := 0
 elif  op(convert(convert(dse, set), list)) then
     k := 2
 else
     k := 1
 end if:
 if ghh<> 0 then
     ee:=0;
     for i in t do
```

```
        if nops(i) = 1 then
            aa:=op(0,i);
            if has(aa,xf) or has(aa,yf) then
                ee:= ee + (coeff(aa,xf)+coeff(aa,yf))*(coeff(op(1,i),x));
             end if:
         else
             bb:=op(0, op(2,i));
             if has(bb,xf) or has(bb,yf) then
                 ee:=ee+((coeff(bb,xf)*op(1,i))+(coeff(bb,yf)*op(1,i)));
              end if:
          end if:
    end do:
    if ee<>0 then
        ff:=ff + c[d]x^(d);
        FF:= FF+ee/(2^(d+1)-2)c[d]x^(d+1);
    else
        ff:=ff + x^(d-1)B[d](x);
        FF:= FF+ x^(d)*B[d](x);
    end if:
end if:
print('f(x)'=sort(ff,x,ascending));
print('F(x)'=sort(FF,x,ascending));
end proc
```

4.4 Application of the computer code

Now we show some examples of functional equations belonging to the class (4.1.1) solved with the use of the above computer code.

Example 4.4.1. *(cf. Example 2 in [28]) Let the pair of functions (F, f) mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfy the equation*

$$F(x+y) - F(x) - F(y) = yf(x) + xf(y) \tag{4.4.1}$$

Observe that equation (4.4.1) is the equation considered by Włodzimierz Fechner and Eszter Gselmann in [11].
INPUT:
> psfe(F(x+y)-F(x)-F(y)-xf(y)-yf(x), [xf(y), yf(x)]);
OUTPUT:

$$f(x) = B_1(x) + c_2x^2$$

$$F(x) = A_1(x) + xB_1(x) + \tfrac{1}{3}c_2x^3$$

where A_1 and B_1 are arbitrary additive functions, and $c_2 \in \mathbb{R}$ is an arbitrary constant.

Example 4.4.2. *(cf. Example 3 in [28]) Let the pair of functions (F, f) mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfy the equation*

$$F(x+y) - F(x) - F(y) = xf(3y) + yf(3x) \tag{4.4.2}$$

INPUT:
> psfe(F(x+y)-F(x)-F(y)-xf(3y)-yf(3x), [xf(3y), yf(3x)]);
OUTPUT:

$$f(x) = B_1(x) + c_2x^2$$

$$F(x) = A_1(x) + 3xB_1(x) + 3c_2x^3$$

where A_1 and B_1 are arbitrary additive functions, and $c_2 \in \mathbb{R}$ is an arbitrary constant.

Example 4.4.3. *(cf. Example 4 in [28]) Let the pair of functions (F, f) mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfy the equation*

$$F(x + y) - F(x) - F(y) = xf(y) \tag{4.4.3}$$

INPUT:
> psfe(F(x+y)-F(x)-F(y)-xf(y), [xf(y)]);
OUTPUT:

$$f(x) = c_1x$$

$$F(x) = A_1(x) + \frac{1}{2}c_1x^2$$

where A_1 is an arbitrary additive functions, and $c_1 \in \mathbb{R}$ is an arbitrary constant.

Example 4.4.4. *(cf. Example 5 in [28]) Let the pair of functions (F, f) mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfy the equation*

$$F(x + y) - F(x) - F(y) = 3xf(2y) - 4yf(3x) \tag{4.4.4}$$

INPUT:
> psfe(F(x+y)-F(x)-F(y)-3xf(2y)+4yf(3x), [3xf(2y), -4yf(3x)]);
OUTPUT:

$$f(x) = c_1x$$

$$F(x) = A_1(x) - 3c_1x^2$$

where A_1 is an arbitrary additive functions, and $c_1 \in \mathbb{R}$ is an arbitrary constant.

Example 4.4.5. *(cf. Example 6 in [28]) Let the pair of functions (F, f) mapping from $\mathbb{R}$ to $\mathbb{R}$ satisfy the equation*

$$F(x + y) - F(x) - F(y) = xf(3y) + yf(3x) + xf(y) + yf(x) + xf(2y) + yf(2x) \tag{4.4.5}$$

INPUT:
> psfe(F(x+y)-F(x)-F(y)-xf(3y)-yf(3x)-xf(y)-yf(x)-xf(2y)-yf(2x), [xf(3y), yf(3x), xf(y), yf(x), xf(2y), yf(2x)]);
OUTPUT:

$$f(x) = B_1(x) + c_2x^2$$

$$F(x) = A_1(x) + 6xB_1(x) + \frac{14}{3}c_2x^3$$

where A_1 and B_1 are arbitrary additive functions, and $c_2 \in \mathbb{R}$ is an arbitrary constant.

Remark 4.4.1. *(cf. Remark 2.3 in [28]) We end the chapter by proposing the future work. The full strength of Lemma 3.1.1 is yet to be utilize in estimating the degree of at least one of the polynomial functions that satisfy a given functional equation defined on a commutative group.*

Therefore, Let G and H be commutative groups (for some results concerning the noncommutative groups see [34] and [3]). Let $\gamma_i \in G$, $\alpha_i, \beta_i, c_j, d_j \in \mathbb{Q}$. Further, let $(F, f) : G \to H$, we intend to find the solutions of a more general functional equation of the type (4.4.6)

$$\sum_{i=1}^{n} \gamma_i F(a_ix + b_iy) = \sum_{j=1}^{m} (\alpha_jx + \beta_jy)f(c_jx + d_jy) \tag{4.4.6}$$

and compare the solutions with the system of equations

$$\begin{cases} \sum_{i=1}^{n} \gamma_i F(a_i x + b_i y) = y f(x) + x f(y) \\ F(x+y) - F(x) - F(y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y) \end{cases} \tag{4.4.7}$$

Let us observe that equation (4.1.1) *is a special case of equation* (4.4.6), *in particular we have* $G = H = \mathbb{R}$, $n = 3, \gamma_1 = a_1 = b_1 = a_2 = b_3 = 1, a_3 = b_2 = 0$ *and* $\gamma_2 = \gamma_3 = -1$.

Chapter 5

Further results on a new class of functional equations satisfied by polynomial functions

5.1 Generalized both sides of the Fechner-Gselmann equation

In this chapter, we consider the following functional equation

$$\sum_{i=1}^{n} \gamma_i F(a_i x + b_i y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y), \tag{5.1.1}$$

for every $x, y \in \mathbb{R}$, $\gamma_i, \alpha_j, \beta_j \in \mathbb{R}$, and $a_i, b_i, c_j, d_j \in \mathbb{Q}$, and its special forms. Thus we continue investigations presented in Chapter 3 (see [25]) where we generalized the left-hand side of Fechner-Gselmann equation and those from Chapter 4 (see [28]) where the right-hand side of the Fechner-Gselmann equation was studied. It turns out that under some assumptions on the parameters involved, the pair (F, f) solving equation (5.1.1) happens to be a pair of polynomial functions.

The idea to study this generalized equation was motivated by the growing number of its particular forms studied by several mathematicians; let us quote here a few of them J. Aczél [1], J. Aczél and M. Kuczma [2], C. Alsina, M. Sablik, and J. Sikorska [4], W. Fechner and E. Gselmann [11], B. Koclęga-Kulpa, T. Szostok and S. Wąsowicz [18], [19] and [20], T. Nadhomi, C. P. Okeke, M. Sablik and T. Szostok [25], and C. P. Okeke and M. Sablik [28]. From their studies, it turns out that these particular forms have real applications.

In this chapter we continue the investigation proposed in Chapter 4 (see Remark 4.4.1). In particular, to obtain the polynomial solutions of equation (5.1.1) and compare the solutions with the solutions of equations

$$\sum_{i=1}^{n} \gamma_i F(a_i x + b_i y) = y f(x) + x f(y), \tag{5.1.2}$$

and

$$F(x+y) - F(x) - F(y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y). \tag{5.1.3}$$

The first of the special forms of (5.1.1) we solved is the functional equation considered by B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [19] namely,

$$F(x) - F(y) = (x - y)[\alpha_1 f(c_1 x + d_1 y) + \cdots + \alpha_m f(c_m x + d_m y)]. \tag{5.1.4}$$

It is worth noting that (5.1.4) stems from a well known quadrature rule used in numerical analysis. Further, we also considered other special forms of (5.1.1) namely,

$$F(y) - F(x) = \frac{1}{y-x}\int_x^y f(t)\,dt = (y-x)\sum_{j=1}^{m} \beta_j f(c_j x + (1-c_j)y), \tag{5.1.5}$$

$$F(y) - F(x) = (y-x)f(x+y), \tag{5.1.6}$$

$$F(x) - F(y) = (x-y)f\left(\tfrac{x+y}{2}\right), \tag{5.1.7}$$

and

$$2F(y) - 2F(x) = (y-x)\left(f\left(\tfrac{x+y}{2}\right) + \tfrac{f(x)+f(y)}{2}\right). \tag{5.1.8}$$

Equation (5.1.5) is the functional equation connected with the Hermite-Hadamard inequality in the class of continuous functions, and it is related to the approximate integration. Note that the quadrature rules of an approximate integration can be obtained by the appropriate specification of the coefficients of (5.1.5). Moreso, equations (5.1.6) and (5.1.7) are variations of Lagrange mean value theorem with many applications in mathematical analysis, computational mathematics and other fields. Finally, equation (5.1.8) stems from the descriptive geometry used for graphical constructions.

In addition we will show that the main results obtained by B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [19] (see Theorem 1 and Theorem 2 in [19]) are special forms of our results. In line with their papers [18] and [20], we would use our method to obtain the polynomial functions connected with the Hermite-Hadamard inequality in the class of continuous functions. Furthermore, we would show that the functional equation considered by J. Aczél in [1] and J. Aczél, M. Kuczma in [2] (cf. Theorem 5 in [2]) are special forms of equation (5.1.1). Moreso, we would show that our method can be used to solve the functional equation arising from the geometric problems considered by C. Alsina, M. Sablik and J. Sikorska in [4].Now observe that equations (5.1.1), (5.1.2) and (5.1.3) are obvious generalization of the equation considered by W. Fechner and E. Gselmann in [11] namely,

$$F(x+y) - F(x) - F(y) = xf(y) + yf(x). \tag{5.1.9}$$

In Chapter 3 (see [25]) T. Nadhomi, C. P. Okeke, M. Sablik and T. Szostok investigated equation (5.1.2) and in Chapter 4 (see [28]) C. P. Okeke and M. Sablik investigated equation (5.1.3). In our works it turns out that under some mild assumption, the pair (F, f) of functions satisfies equations (5.1.2), and (5.1.3) are polynomial functions, and in some important cases, just the usual polynomials (even though we assume no regularity of solutions a priori).

While proving our main results in Chapter 3 (see [25]) and Chapter 4 (see [28]), we observed that the behaviour of solutions depends on the sequences $(L_k)_{k\in\mathbb{N}\cup\{0\}}$ and $(R_k)_{k\in\mathbb{N}\cup\{0\}}$ given by

$$L_k = \sum_{i=1}^{n} \gamma_i (a_i + b_i)^{k+1}, \tag{5.1.10}$$

and

$$R_k = \sum_{j=1}^{m} (\alpha_j + \beta_j)(c_j + d_j)^k, \tag{5.1.11}$$

respectively, for all $k \in \mathbb{N}\cup\{0\}$.

5.1.1 Main Results

We begin by showing that in general equation (5.1.1) has polynomial functions as solutions. To this name, rewrite (5.1.1) in the following form:

$$\sum_{(a_i,b_i)} \gamma_i F(a_i x + b_i y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y),$$

which allows us to write the left-hand side in the form

$$\sum_{\{(a_i,b_i):a_i\neq 0\neq b_i\}} \gamma_i F(a_i x + b_i y) + \sum_{\{(a_i,0):a_i\neq 0\}} \gamma_i F(a_i x) + \sum_{\{(0,b_i):0\neq b_i\}} \gamma_i F(b_i y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y). \tag{5.1.12}$$

We excluded above the summands where $a_i = 0 = b_i$. Such a summand can be omitted, indeed. Namely suppose that $a_n = b_n = 0$. Let F be a solution of (5.1.1), and assume that $n \geqslant 2$ (otherwise the whole problem becomes trivial). If we put $x = y = 0$ in (5.1.1) then we get

$$\left(\sum_{i=1}^{n-1} \gamma_i + \gamma_n\right) F(0) = 0. \tag{5.1.13}$$

From (5.1.13) we infer that either $F(0) = 0$ or $\sum_{i=1}^{n-1} \gamma_i + \gamma_n = 0$. In the former the constant $F(0)$ disappears, and the left hand side of (5.1.1) satisfies our assumptions. In the latter case, if moreover $\gamma_n = 0$, the situation is analogous. Let us consider therefore the case

$$\left(\sum_{i=1}^{n-1} \gamma_i + \gamma_n = 0\right) \wedge (\gamma_n \neq 0)\,.$$

Observe that $\sum_{i=1}^{n-1} \gamma_i \neq 0$, and

$$\gamma_n F(0) = \sum_{i=1}^{n-1} \gamma_i \frac{\gamma_n}{\sum_{i=1}^{n-1} \gamma_i} F(0).$$

Hence (5.1.1) may written in the form

$$\sum_{i=1}^{n-1} \gamma_i \left(F(a_i x + b_i y) + \frac{\gamma_n}{\sum_{i=1}^{n-1} \gamma_i} F(0) \right) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y).$$

Substituting $\tilde{F}(z) := F(z) + \frac{\gamma_n}{\sum_{i=1}^{n-1} \gamma_i} F(0)$ we obtain

$$\sum_{i=1}^{n-1} \gamma_i \tilde{F}(a_i x + b_i y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y),$$

where $(a_i, b_i) \neq (0,0), i \in \{1, \cdots, n-1\}$.

From (5.1.12) we see that there are three essential groups of terms on left-hand side, the first group contains the summands containing values of F at the points $a_i x + b_i y$, where $a_i \neq 0 \neq b_i$. The second group contains those summands in which $b_i = 0$, and the third one those in which $a_i = 0$. We saw in Chapter 3 (see Example 3.2.1), that if the first and the third groups are empty, and the second one consists of two pairs $(1, 0)$ and $(-1, 0)$ and the corresponding γ's are 1 and -1, then arbitrary even function F yields a solution to (5.1.12), together with $f = 0$. Thus in general there is no chance that we obtain polynomiality of F. However, a closer look shows that we can state some positive claims.

Namely, rewrite again (5.1.12) in the form

$$\sum_{i\in I_1} \varphi_i(a_ix + b_iy) + \varphi_I(x) + \varphi_{II}(y) = \sum_{j=1}^{m} (\alpha_jx + \beta_jy)f(c_jx + d_jy), \tag{5.1.14}$$

and assume that there is a $j \in \{1, \ldots, m\}$ such that

$$\det \begin{pmatrix} \alpha_j & \beta_j \\ c_j & d_j \end{pmatrix} \neq 0, \qquad \text{and} \qquad d_j \neq 0. \tag{5.1.15}$$

Then it is possible to perform the change of variables $c_jx + d_jy = z$, $x = w$. It remains to express (x, y) in terms of (z, w) and to rewrite (5.1.14) in the form

$$Cf(z)w = P_{f,\varphi_I,\varphi_{II},\{\varphi_i:i\in I_1\}}(z, w),$$

where $P_{f,\varphi_I,\varphi_{II},\{\varphi_i:i\in I_1\}}$ is a polynomial function in z and w, with coefficients depending on $f, \varphi_I, \varphi_{II}$ and $\varphi_i, i \in I_1$. We can apply Lemma 3.1.1 to get the polynomiality of f.

Thus the right-hand side of (5.1.12) is a polynomial in x and y, say $Q(x, y)$. In other words we have

$$\sum_{i\in I_1} \varphi_i(a_ix + b_iy) + \varphi_I(x) + \varphi_{II}(y) = Q(x, y). \tag{5.1.16}$$

If $\varphi_I \neq 0$ then we may rewrite the above in the form

$$\varphi_I(x) = -\sum_{i\in I_1} \varphi_i(a_ix + b_iy) - \varphi_{II}(y) + Q(x, y), \tag{5.1.17}$$

whence by induction we obtain that φ_I is a polynomial function in x. Actually, assuming that Q is a polynomial of order say k in x, we apply the operator Δ, $k+1$ times to both sides of equation (5.1.17). We get

$$\Delta_{h_1,\ldots,h_{k+1}}\varphi_I(x) = -\sum_{i\in I_1} \Delta_{h_1,\ldots,h_{k+1}}\varphi_i(a_ix + b_iy) + \tilde{\varphi}_{II}(y) + R(y), \tag{5.1.18}$$

where R is a polynomial function in y (which remains after annihilating the x part of $Q(x, y)$). Now, denoting $\Delta_{h_1,\ldots,h_{k+1}}\varphi_I$ by $\overline{\varphi}$ and $\Delta_{h_1,\ldots,h_{k+1}}\varphi_i$ by f_i, as well as $\tilde{\varphi}_{II}(y) + R(y)$ by $\overline{f}$, we get the equation from the Székelyhidi's result(see (2.2.2)). We use it to infer that $\overline{\varphi}$ is a polynomial function, whence polynomiality of $\varphi_I = F$ follows.

If we knew that $\varphi_I(x) = DF(x)$ for some constants D then we are done. Analogously, if $\varphi_{II}(y) \neq 0$ with similar argument as above we get polynomiality of $\varphi_{II} = F$. But even if $\varphi_I(x) \neq DF(x)$ or $\varphi_{II}(y) \neq EF(y)$ for some constants E and D, we can still look at the first summand on the left-hand side of (5.1.16). Then it is enough to see whether the first sum is non zero and admits $z = a_{i_0}x + b_{i_0}y, w = x$ for some i_0 and to rewrite (5.1.16) in the form

$$\varphi_{i_0}(z) = -\sum_{i\neq i_0} \varphi_i(e_iz + f_iw) - \varphi_I(w) - \varphi_{II}(g_iz + h_iw) + \tilde{Q}(z, w), \tag{5.1.19}$$

and hence we see, similarly as before, that F has to be a polynomial function.

Therefore, it is enough to assume that

1. There exist a $j \in \{1, \cdots, m\}$ such that (5.1.15) holds and

2. $\varphi_I = DF$ or,

3. $\varphi_{II} = EF$ or,

4. for some $i_0 \in \{1, \cdots, n\}$ we have $a_{i_0} \neq 0 \neq b_{i_0}$,

to get polynomiality of both f and F.

Having a result of this kind, it is now enough to assume that the pair of functions (F, f) satisfying equation (5.1.1) are monomials. However, from equations (5.1.2) and (5.1.3), we may assume that a characteristic feature of equation (5.1.1) is the dependence of the existence of solutions on the sequences given by (5.1.10) and (5.1.11), respectively. Hence, we proceed to next theorem.

Theorem 5.1.1. *(cf. Theorem 2.1 in [26]) Suppose* $\gamma_i, \alpha_j, \beta_j \in \mathbb{R}$, $a_i, b_i, c_j, d_j \in \mathbb{Q}, i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$. *Let* $(L_k)_{k \in \mathbb{N} \cup \{0\}}$ *and* $(R_k)_{k \in \mathbb{N} \cup \{0\}}$ *be defined by (5.1.10) and (5.1.11) respectively. Assume that* $L_k, R_k \neq 0$ *for some* $k \in \mathbb{N} \cup \{0\}$*, and equation (5.1.1) is satisfied by the pair* $(F, f) : \mathbb{R} \longrightarrow \mathbb{R}$ *of monomial functions of order* $k + 1$ *and* k*, respectively.*

(i) if $k = 0$ *then* $f = 0 = F$ *or* $f = A_0 \neq 0$ *and* $F(x) = \frac{R_0}{L_0} A_0 x$*; in the latter case necessarily*

$$\tfrac{R_0}{L_0} \sum_{i=1}^{n} \gamma_i a_i = \sum_{j=1}^{m} \alpha_j, \tag{5.1.20}$$

and

$$\tfrac{R_0}{L_0} \sum_{i=1}^{n} \gamma_i b_i = \sum_{j=1}^{m} \beta_j. \tag{5.1.21}$$

(ii) if $k \neq 0$ *then either* $f = F = 0$ *is the only solution of (5.1.1), or* f *is an arbitrary additive function while* F *is given by* $F(x) = \frac{R_k}{L_k} x f(x)$, $x \in \mathbb{R}$*, when the below equations holds*

$$\tfrac{R_k}{L_k} \sum_{i=1}^{n} \gamma_i a_i^{k+1} = \sum_{j=1}^{m} \alpha_j c_j^k, \tag{5.1.22}$$

$$\tfrac{R_k}{L_k} \sum_{i=1}^{n} \gamma_i b_i^{k+1} = \sum_{j=1}^{m} \beta_j d_j^k, \tag{5.1.23}$$

and

$$\tfrac{R_k}{L_k} \sum_{i=1}^{n} \binom{k+1}{p} \gamma_i a_i^p b_i^{k+1-p} = \sum_{j=1}^{m} \binom{k}{p} \beta_j c_j^p d_j^{k-p} + \sum_{j=1}^{m} \binom{k}{p-1} \alpha_j c_j^{p-1} d_j^{k+1-p}, \tag{5.1.24}$$

for each $p \in \{1, \cdots, k\}$*. Furthermore, for non-trivial* f *we see that either*

(a) $\sum_{j=1}^{m} \beta_j c_j^p d_j^{k-p} = \sum_{j=1}^{m} \alpha_j c_j^{p-1} d_j^{k+1-p}$ *for each* $p \in \{1, \cdots, k\}$*, and* f *is an arbitrary* k*-monomial function, or*

(b) $\sum_{j=1}^{m} \beta_j c_j^p d_j^{k-p} \neq \sum_{j=1}^{m} \alpha_j c_j^{p-1} d_j^{k+1-p}$ *for each* $p \in \{1, \cdots, k\}$*, and* f *is necessarily a continuous monomial function of order* k *and so is* F *of order* $k + 1$.

Proof. Suppose that $k = 0$. Then $f = \text{const} = A_0$ and F is additive. Putting $x = y$ in (5.1.1) we get

$$L_0 F(x) = \sum_{j=1}^{m} (\alpha_j + \beta_j) x A_0,$$

i.e.

$$F(x) = \tfrac{R_0}{L_0} x A_0 = Cx, \tag{5.1.25}$$

for every $x \in \mathbb{R}$, since $L_0, R_0 \neq 0$, thus F is a continuous function. Substituting (5.1.25) into (5.1.1) we obtain

$$C\left(\sum_{i=1}^{n} \gamma_i a_i\right) x + C\left(\sum_{i=1}^{n} \gamma_i b_i\right) y = A_0\left(\sum_{j=1}^{m} \alpha_j\right) x + A_0\left(\sum_{j=1}^{m} \beta_j\right) y,$$

for all $x, y \in \mathbb{R}$, whence formulae (5.1.20) and (5.1.21) easily follow. Observe that it is impossible to have

$$\sum_{i=1}^{n} \gamma_i a_i = \sum_{i=1}^{n} \gamma_i b_i = 0. \tag{5.1.26}$$

Indeed, in such a case $L_0 = 0$, which contradicts our assumption.

Suppose that $k = 1$, we obtain that $f = A_1$ is additive and $F = B_2^*$ is a quadratic function, or in other words diagonalization of a biadditive function. Putting $x = y$ in (5.1.1) we obtain (taking into account the rational homogeneity of f and F)

$$L_1 B_2^*(x) = \left(\sum_{j=1}^{m} (\alpha_j + \beta_j)(c_j + d_j)\right) x A_1(x),$$

whence,

$$L_1 F(x) = R_1 x A_1(x),$$

for every $x \in \mathbb{R}$. Keeping in mind that $L_1 \neq 0$ and denoting $E_1 = \frac{R_1}{L_1}$ we get

$$F(x) = E_1 x A_1(x),$$

for every $x \in \mathbb{R}$. Substituting the above into (5.1.1) we obtain

$$\begin{aligned} E_1\left(\sum_{i=1}^{n} \gamma_i a_i^2\right) x A_1(x) + E_1\left(\sum_{i=1}^{n} \gamma_i b_i^2\right) y A_1(y) \\ + E_1\left(\sum_{i=1}^{n} \gamma_i a_i b_i\right) x A_1(y) + E_1\left(\sum_{i=1}^{n} \gamma_i a_i b_i\right) y A_1(x) \\ = \left(\sum_{j=1}^{m} \alpha_j c_j\right) x A_1(x) + \left(\sum_{j=1}^{m} \beta_j d_j\right) y A_1(y) \\ + \left(\sum_{j=1}^{m} \alpha_j d_j\right) x A_1(y) + \left(\sum_{j=1}^{m} \beta_j c_j\right) y A_1(x). \end{aligned} \tag{5.1.27}$$

Comparing the terms with the same degrees we obtain

$$\left(E_1\left(\sum_{i=1}^{n} \gamma_i a_i^2\right) - \left(\sum_{j=1}^{m} \alpha_j c_j\right)\right) x A_1(x) = 0, \tag{5.1.28}$$

$$\left(E_1\left(\sum_{i=1}^{n} \gamma_i b_i^2\right) - \left(\sum_{j=1}^{m} \beta_j d_j\right)\right) y A_1(y) = 0, \tag{5.1.29}$$

$$\begin{aligned} \left(E_1\left(\sum_{i=1}^{n} \gamma_i a_i b_i\right) - \left(\sum_{j=1}^{m} \alpha_j d_j\right)\right) x A_1(y) \\ = \left(\left(\sum_{j=1}^{m} \beta_j c_j\right) - E_1\left(\sum_{i=1}^{n} \gamma_i a_i b_i\right)\right) y A_1(x). \end{aligned} \tag{5.1.30}$$

Observe that (5.1.28) holds if either $A_1 = 0$ or

$$E_1\left(\sum_{i=1}^{n} \gamma_i a_i^2\right) = \sum_{j=1}^{m} \alpha_j c_j. \tag{5.1.31}$$

Similarly, (5.1.29) holds if either $A_1 = 0$ or

$$E_1\left(\sum_{i=1}^{n}\gamma_i b_i^2\right) = \sum_{j=1}^{m}\beta_j d_j. \tag{5.1.32}$$

Finally, (5.1.30) holds if either $A_1 = 0$ or,

$$2E_1\left(\sum_{i=1}^{n}\gamma_i a_i b_i\right) = \sum_{j=1}^{m}\beta_j c_j + \sum_{j=1}^{m}\alpha_j d_j. \tag{5.1.33}$$

Now if $A_1 = 0$ then $F = 0$. Let us consider the non-zero solutions of (5.1.1). Then all equations (5.1.31), (5.1.32) and (5.1.33) hold. Note that it is impossible that

$$\sum_{i=1}^{n}\gamma_i a_i^2 = \sum_{i=1}^{n}\gamma_i b_i^2 = \sum_{i=1}^{n}\gamma_i a_i b_i = 0.$$

In fact, in such a situation we would have $L_1 = 0$, which contradicts our assumption. Moreover, substituting (5.1.33) into (5.1.30) we get

$$\left(\sum_{j=1}^{m}\beta_j c_j - \sum_{j=1}^{m}\alpha_j d_j\right)(xA_1(y) - yA_1(x)) = 0, \tag{5.1.34}$$

for all $x, y \in \mathbb{R}$. From (5.1.34) we see that either

$$\sum_{j=1}^{m}\beta_j c_j = \sum_{j=1}^{m}\alpha_j d_j,$$

which leads to a situation where A_1 can be an arbitrary (in particular discontinuous) additive function and we get that the pair (F, f) is a solution of (5.1.1), or

$$yA_1(x) = xA_1(y),$$

for all $x, y \in \mathbb{R}$. Putting $y = 1$ in the above equation we have

$$A_1(x) = xA_1(1),$$

for every $x \in \mathbb{R}$, hence f and F are continuous.

Now, let us pass to the situation where $k \geqslant 2$. In general, if $k \geqslant 2$ and the pair (F, f) satisfies (5.1.1) then

$$f(x) = A_k^*(x),$$

for every $x \in \mathbb{R}$, and hence

$$F(x) = \frac{R_k}{L_k}xA_k^*(x),$$

for every $x \in \mathbb{R}$. Denote $E_k = \frac{R_k}{L_k}$, we can write (5.1.1) as

$$E_k\sum_{i=1}^{n}\gamma_i(a_i x + b_i y)A_k^*(a_i x + b_i y) = \sum_{j=1}^{m}(\alpha_j x + \beta_j y)A_k^*(c_j x + d_j y),$$

or

$$\begin{aligned} E_k\sum_{i=1}^{n}\gamma_i&\left[a_i x\left(\sum_{p=0}^{k}\binom{k}{p}a_i^p b_i^{k-p}A_k(x^p, y^{k-p})\right)\right.\\ &\left.+\, b_i y\left(\sum_{p=0}^{k}\binom{k}{p}a_i^p b_i^{k-p}A_k(x^p, y^{k-p})\right)\right]\\ &= \sum_{j=1}^{m}(\alpha_j x + \beta_j y)\left(\sum_{p=0}^{k}\binom{k}{p}c_j^p d_j^{k-p}A_k(x^p, y^{k-p})\right), \end{aligned}$$

whence

$$\begin{aligned}
&E_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{p=0}^{k} \binom{k}{p} a_i^{p+1} b_i^{k-p} x A_k(x^p, y^{k-p})\right)\right] \\
&+E_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{p=0}^{k} \binom{k}{p} a_i^{p} b_i^{k+1-p} y A_k(x^p, y^{k-p})\right)\right] \\
&\quad = \sum_{j=1}^{m} \alpha_j \left(\sum_{p=0}^{k} \binom{k}{p} c_j^{p} d_j^{k-p} x A_k(x^p, y^{k-p})\right) \\
&\quad + \sum_{j=1}^{m} \beta_j \left(\sum_{p=0}^{k} \binom{k}{p} c_j^{p} d_j^{k-p} y A_k(x^p, y^{k-p})\right),
\end{aligned}$$

or

$$\begin{aligned}
&E_k \sum_{i=1}^{n} \gamma_i a_i^{k+1} x A_k^*(x) + E_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{p=0}^{k-1} \binom{k}{p} a_i^{p+1} b_i^{k-p} x A_k(x^p, y^{k-p})\right)\right] \\
&+E_k \sum_{i=1}^{n} \gamma_i b_i^{k+1} y A_k^*(y) + E_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{p=1}^{k} \binom{k}{p} a_i^{p} b_i^{k+1-p} y A_k(x^p, y^{k-p})\right)\right] \\
&\quad = \sum_{j=1}^{m} \alpha_j c_j^k x A_k^*(x) + \sum_{j=1}^{m} \alpha_j \left(\sum_{p=0}^{k-1} \binom{k}{p} c_j^{p} d_j^{k-p} x A_k(x^p, y^{k-p})\right) \\
&\quad + \sum_{j=1}^{m} \beta_j d_j^k y A_k^*(y) + \sum_{j=1}^{m} \beta_j \left(\sum_{p=1}^{k} \binom{k}{p} c_j^{p} d_j^{k-p} y A_k(x^p, y^{k-p})\right),
\end{aligned}$$

or

$$\begin{aligned}
&E_k \sum_{i=1}^{n} \gamma_i a_i^{k+1} x A_k^*(x) + E_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{p=1}^{k} \binom{k}{p-1} a_i^{p} b_i^{k+1-p} x A_k(x^{p-1}, y^{k+1-p})\right)\right] \\
&+E_k \sum_{i=1}^{n} \gamma_i b_i^{k+1} y A_k^*(y) + E_k \sum_{i=1}^{n} \gamma_i \left[\left(\sum_{p=1}^{k} \binom{k}{p} a_i^{p} b_i^{k+1-p} y A_k(x^p, y^{k-p})\right)\right] \\
&= \sum_{j=1}^{m} \alpha_j c_j^k x A_k^*(x) + \sum_{j=1}^{m} \alpha_j \left(\sum_{p=1}^{k} \binom{k}{p-1} c_j^{p-1} d_j^{k+1-p} x A_k(x^{p-1}, y^{k+1-p})\right) \\
&\quad + \sum_{j=1}^{m} \beta_j d_j^k y A_k^*(y) + \sum_{j=1}^{m} \beta_j \left(\sum_{p=1}^{k} \binom{k}{p} c_j^{p} d_j^{k-p} y A_k(x^p, y^{k-p})\right),
\end{aligned}$$

for all x, $y \in \mathbb{R}$. Comparing terms of equal degrees we have the following equations

$$\left(E_k \sum_{i=1}^{n} \gamma_i a_i^{k+1} - \sum_{j=1}^{m} \alpha_j c_j^k\right) x A_k^*(x) = 0, \tag{5.1.35}$$

$$\left(E_k \sum_{i=1}^{n} \gamma_i b_i^{k+1} - \sum_{j=1}^{m} \beta_j d_j^k\right) y A_k^*(y) = 0, \tag{5.1.36}$$

$$\begin{aligned}
&\left(E_k \sum_{i=1}^{n} \binom{k}{p-1} \gamma_i a_i^p b_i^{k+1-p} - \sum_{j=1}^{m} \binom{k}{p-1} \alpha_j c_j^{p-1} d_j^{k+1-p}\right) x A_k(x^{p-1}, y^{k+1-p}) \\
&\quad = \left(\sum_{j=1}^{m} \binom{k}{p} \beta_j c_j^p d_j^{k-p} - E_k \sum_{i=1}^{n} \binom{k}{p} \gamma_i a_i^p b_i^{k+1-p}\right) y A_k(x^p, y^{k-p}),
\end{aligned} \tag{5.1.37}$$

for $p \in \{1, \cdots, k\}$ and all x, $y \in \mathbb{R}$. Now, we observe that if (5.1.35) holds then either $A_k = 0$ or

$$E_k \sum_{i=1}^{n} \gamma_i a_i^{k+1} = \sum_{j=1}^{m} \alpha_j c_j^k. \tag{5.1.38}$$

Similarly, (5.1.36) holds if either $A_k = 0$ or

$$E_k \sum_{i=1}^{n} \gamma_i b_i^{k+1} = \sum_{j=1}^{m} \beta_j d_j^k. \tag{5.1.39}$$

Finally, (5.1.37) holds if either $A_k = 0$ or

$$E_k \sum_{i=1}^{n} \binom{k+1}{p} \gamma_i a_i^p b_i^{k+1-p} = \sum_{j=1}^{m} \binom{k}{p} \beta_j c_j^p d_j^{k-p} + \sum_{j=1}^{m} \binom{k}{p-1} \alpha_j c_j^{p-1} d_j^{k+1-p}, \tag{5.1.40}$$

for $p \in \{1, \cdots, k\}$. Assume from now on that we are interested in nontrivial solutions of (5.1.1), that is when $A_k \neq 0$ and equations (5.1.38), (5.1.39) and (5.1.40) holds. Observe that is impossible to have

$$\sum_{i=1}^{n} \gamma_i a_i^{k+1} = \sum_{i=1}^{n} \gamma_i b_i^{k+1} = \sum_{i=1}^{n} \binom{k+1}{p} \gamma_i a_i^p b_i^{k+1-p} = 0,$$

for each $p \in \{1, \cdots, k\}$. Indeed, in such a case we would have that $L_k = 0$, which contradicts our assumption. Now, substituting (5.1.40) into (5.1.37)

$$\left(\sum_{j=1}^{m} \beta_j c_j^p d_j^{k-p} - \sum_{j=1}^{m} \alpha_j c_j^{p-1} d_j^{k+1-p} \right) \left(x A_k(x^{p-1}, y^{k+1-p}) - y A_k(x^p, y^{k-p}) \right) = 0, \tag{5.1.41}$$

for $p \in \{1, \cdots, k\}$ and all x, $y \in \mathbb{R}$. Now from (5.1.41) we see that either

$$\sum_{j=1}^{m} \beta_j c_j^p d_j^{k-p} = \sum_{j=1}^{m} \alpha_j c_j^{p-1} d_j^{k+1-p},$$

for each $p \in \{1, \cdots, k\}$, which leads to a situation where A_k can be an arbitrary additive function and we get that the pair (F, f) is a solution of (5.1.1) or

$$x A_k(x^{p-1}, y^{k+1-p}) = y A_k(x^p, y^{k-p}), \tag{5.1.42}$$

for $p \in \{1, \cdots, k\}$ and all x, $y \in \mathbb{R}$. Now, using (5.1.42) for $p \in \{1, \cdots, k\}$ we arrive at

$$y^k A_k^*(x) = y^{k-1} \left[y A_k^*(x) \right] = y^{k-1} \left[x A_k(x^{k-1}, y) \right] = \cdots = x^k A_k^*(y),$$

for every x, $y \in \mathbb{R}$, in other words, putting $y = 1$ we obtain

$$A_k^*(x) = A_k^*(1) x^k, \tag{5.1.43}$$

for every $x \in \mathbb{R}$, which means that A_k is continuous for $k \geqslant 2$ and thus ends the proof. □

Remark 5.1.1. *(cf. Remark 2.1 in [26]) We note here that in equations (5.1.1) and (5.1.10), if $f = 0$ and $k \in \mathbb{N} \cup \{0\}$ with*

$$L_k = \sum_{i=1}^{n} \gamma_i (a_i + b_i)^{k+1} = 0,$$

then F is not necessarily equal to zero. Of course this does not contradict Theorem 5.1.1 because $L_k \neq 0$. Therefore, we state the below propositions.

Proposition 5.1.1. *(cf. Proposition 2.2 in [26]) Let $\gamma_i \in \mathbb{R}$, $a_i, b_i \in \mathbb{Q}$, $i \in \{1, \cdots, n\}$. Let $(L_k)_{k \in \mathbb{N} \cup \{0\}}$ be defined by (5.1.10). Assume that $k = 0$ such that*

$$L_0 = \sum_{i=1}^{n} \gamma_i (a_i + b_i) = 0, \tag{5.1.44}$$

holds then either $f = F = 0$ is the only solution of (5.1.1) or,

(a) *If* $f = 0$ *and* $F = const = A_0$ *where* A_0 *is any real number is the solution to* (5.1.1) *then*

$$\sum_{i=1}^{n} \gamma_i = 0.$$

(b) *If* $f = 0$ *and* $F = A_1$ *is an additive function is the solution to* (5.1.1) *then*

$$\sum_{i=1}^{n} \gamma_i a_i = \sum_{i=1}^{n} \gamma_i b_i = 0.$$

Proof. Suppose that (5.1.44) holds. Let $f = 0$ and $F = const = A_0$ where A_0 is any real number. Substituting this in equation (5.1.1) we have

$$\sum_{i=1}^{n} \gamma_i A_0 = 0,$$

this holds if either $A_0 = 0$ or $\sum_{i=1}^{n} \gamma_i = 0$. Now for non trivial solutions of (5.1.1) we have that $A_0 \neq 0$ and $\sum_{i=1}^{n} \gamma_i = 0$.

Finally, suppose that (5.1.44) holds. Let $f = 0$ and $F = A_1$ be additive. Substituting this in equation (5.1.1) we get (taking into account the rational homogeneity of A_1)

$$\sum_{i=1}^{n} \gamma_i A_1(a_i x + b_i y) = 0,$$

i.e.

$$\sum_{i=1}^{n} \gamma_i a_i A_1(x) + \sum_{i=1}^{n} \gamma_i b_i A_1(y) = 0.$$

Comparing terms of the same degree on both sides of the above equation, we obtain

$$\sum_{i=1}^{n} \gamma_i a_i A_1(x) = 0,$$

for all $x \in \mathbb{R}$, and symmetrically

$$\sum_{i=1}^{n} \gamma_i b_i A_1(y) = 0,$$

for all $y \in \mathbb{R}$. Both of these equations hold if either $A_1 = 0$ or $\sum_{i=1}^{n} \gamma_i a_i = \sum_{i=1}^{n} \gamma_i b_i = 0$. Now for non trivial solutions of (5.1.1) we have that $A_1 \neq 0$ and $\sum_{i=1}^{n} \gamma_i a_i = \sum_{i=1}^{n} \gamma_i b_i = 0$. □

Proposition 5.1.2. *(cf. Proposition 2.3 in [26]) Let* $\gamma_i \in \mathbb{R}$, $a_i, b_i \in \mathbb{Q}$, $i \in \{1, \cdots, n\}$. *Let* $(L_k)_{k \in \mathbb{N} \cup \{0\}}$ *be defined by* (5.1.10). *Assume that* $k \in \mathbb{N}$ *such that*

$$L_k = \sum_{i=1}^{n} \gamma_i (a_i + b_i)^{k+1} = 0, \tag{5.1.45}$$

holds then either $f = F = 0$ *is the only solution of* (5.1.1) *or,* $f = 0$ *and* $F = A^*_{k+1}$ *is an arbitrary* $k+1$ *additive function when*

$$\sum_{i=1}^{n} \binom{k+1}{p} \gamma_i a_i^p b_i^{k+1-p} = 0,$$

for each $p \in \{0, \cdots, k+1\}$.

Remark 5.1.2. *(cf. Remark 2.2 in [26]) We note that if* $f = 0$, $k = 0$, *and* $\sum_{i=1}^{n} \gamma_i = 0$ *then* $F = A_0$ *where* A_0 *is any real number is also a solution to* (5.1.1).

Remark 5.1.3. *(cf. Remark 2.3 in [26]) Since we are interested in the pair* (F, f) *of polynomial functions that satisfies* (5.1.1)*, thus, we mention here that assumptions* (5.1.44)*,* (5.1.45) *and Remark 5.1.2 are essential when* $f = 0$*. Therefore, if* $f = 0$ *and* $k \in \mathbb{N} \cup \{0\}$ *with* $L_k \neq 0$ *then* $f = F = 0$ *is the only solution to* (5.1.1).

5.2 Applications

5.2.1 Functional equations connected with quadrature rules

Now, we show that the main results obtained by B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [19] (see Theorem 1 and Theorem 2 in [19]) are indeed special forms of our results.

Theorem 5.2.1. *(cf. Theorem 1 in [19] and Theorem 2.4 in [26]) The functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\begin{aligned} 8F(y) - 8F(x) = yf(x) + 3yf\left(\tfrac{x+2y}{3}\right) + 3yf\left(\tfrac{2x+y}{3}\right) + \\ yf(y) - xf(x) - 3xf\left(\tfrac{x+2y}{3}\right) - 3xf\left(\tfrac{2x+y}{3}\right) - xf(y) \end{aligned} \tag{5.2.1}$$

for $x, y \in \mathbb{R}$, if and only if

$$f(x) = ax^3 + bx^2 + cx + d$$

and

$$F(x) = \tfrac{1}{4}ax^4 + \tfrac{1}{3}bx^3 + \tfrac{1}{2}cx^2 + dx + e$$

for all $x \in \mathbb{R}$ and $a, b, c, d, e \in \mathbb{R}$.

Proof. Suppose that the pair (F, f) satisfies equation (5.2.1), then putting $y = x + y$ in the equation we get

$$8F(x+y) - 8F(x) = yf(x) + 3yf\left(\tfrac{3x+2y}{3}\right) + 3yf\left(\tfrac{3x+y}{3}\right) + yf(x+y). \tag{5.2.2}$$

Now rearranging (5.2.2) in the form

$$yf(x) + 8F(x) = 8F(x+y) - 3yf\left(\tfrac{3x+2y}{3}\right) - 3yf\left(\tfrac{3x+y}{3}\right) - yf(x+y), \tag{5.2.3}$$

and applying Lemma 3.1.1, we get $I_{0,0} = \{(id, id)\}$, $I_{0,1} = \{(id, \frac{2}{3}id), (id, \frac{1}{3}id), (id, id)\}$, $\psi_{0,0,(id,id)} = F$, $\psi_{0,1,(id,\frac{2}{3}id)} = -f$, $\psi_{0,1,(id,\frac{1}{3}id)} = -f$, $\psi_{0,1,(id,id)} = -f$, $\varphi_0 = F$, $\varphi_1 = f$. We also have $K_0 = I_{0,0}$, $K_1 = I_{0,1}$, and $K_0 \cup K_1 = \{(id, \frac{2}{3}id), (id, \frac{1}{3}id), (id, id)\}$. Therefore, $\varphi_1 = f$ is a polynomial function of degree at most $m = 5$ i.e.

$$m = \mathrm{card}(\mathrm{K}_0 \cup \mathrm{K}_1) + \mathrm{card}(\mathrm{K}_1) - 1 = 3 + 3 - 1 = 5.$$

Observe that (5.2.1) is a special form of (5.1.1), thus we have that F is a polynomial function. Now we check conditions of Theorem 5.1.1. If $k = 0$, then $f(x) = d$, for some constant $d \in \mathbb{R}$ and all $x \in \mathbb{R}$, further from (5.1.20) and (5.1.21) we have

$$\tfrac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i a_i = \sum_{j=1}^{8} \alpha_j$$

and

$$\tfrac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i b_i = \sum_{j=1}^{8} \beta_j.$$

Hence, $\frac{R_0}{L_0} = 1$, and consequently $F(x) = dx$ for some constant $d \in \mathbb{R}$ and all $x \in \mathbb{R}$. If $k = 1$ we get,

$$\tfrac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i a_i^2 = \sum_{j=1}^{8} \alpha_j c_j,$$

$$\tfrac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i b_i^2 = \sum_{j=1}^{8} \beta_j d_j,$$

and

$$\tfrac{R_1}{L_1}\sum_{i=1}^{2}2\gamma_i a_i b_i = 0 = \sum_{j=1}^{8}\beta_j c_j + \sum_{j=1}^{8}\alpha_j d_j,$$

thus, $\frac{R_1}{L_1} = \frac{1}{2}$ and also,

$$\sum_{j=1}^{8}\beta_j c_j = 4 \neq -4 = \sum_{j=1}^{8}\alpha_j d_j$$

then by Theorem 5.1.1, we infer that the monomial functions F, f are continuous, therefore $f(x) = cx$ and $F(x) = \frac{1}{2}cx^2$ for some constant $c \in \mathbb{R}$ and all $x \in \mathbb{R}$. Now let $k = 2$ then we have,

$$\tfrac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i a_i^3 = \sum_{j=1}^{8}\alpha_j c_j^2,$$

$$\tfrac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i b_i^3 = \sum_{j=1}^{8}\beta_j d_j^2,$$

and

$$\tfrac{R_2}{L_2}\sum_{i=1}^{2}\binom{3}{p}\gamma_i a_i^p b_i^{3-p} = 0 = \sum_{j=1}^{8}\binom{2}{p}\beta_j c_j^p d_j^{2-p} + \sum_{j=1}^{8}\binom{2}{p-1}\alpha_j c_j^{p-1} d_j^{3-p},$$

for each $p \in \{1, 2\}$. Hence, $\frac{R_2}{L_2} = \frac{1}{3}$ and also,

$$\sum_{j=1}^{8}\beta_j c_j^p d_j^{2-p} = \tfrac{8}{3} \neq -\tfrac{8}{3} = \sum_{j=1}^{8}\alpha_j c_j^{p-1} d_j^{3-p},$$

for each $p \in \{1, 2\}$. By Theorem 5.1.1 we infer that the monomial functions F, f are continuous, therefore $f(x) = bx^2$ and $F(x) = \frac{1}{3}bx^3$ for some constant $b \in \mathbb{R}$ and all $x \in \mathbb{R}$. If $k = 3$, then we obtain,

$$\tfrac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i a_i^4 = \sum_{j=1}^{8}\alpha_j c_j^3,$$

$$\tfrac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i b_i^4 = \sum_{j=1}^{8}\beta_j d_j^3,$$

and

$$\tfrac{R_3}{L_3}\sum_{i=1}^{2}\binom{4}{p}\gamma_i a_i^p b_i^{4-p} = 0 = \sum_{j=1}^{8}\binom{3}{p}\beta_j c_j^p d_j^{3-p} + \sum_{j=1}^{8}\binom{3}{p-1}\alpha_j c_j^{p-1} d_j^{4-p},$$

for each $p \in \{1, 2, 3\}$. Hence, $\frac{R_3}{L_3} = \frac{1}{4}$ and also,

$$\sum_{j=1}^{8}\beta_j c_j^p d_j^{3-p} = 2 \neq -2 = \sum_{j=1}^{8}\alpha_j c_j^{p-1} d_j^{4-p},$$

for each $p \in \{1, 2, 3\}$. Again by Theorem 5.1.1 we infer that the monomial functions (F, f) are continuous, therefore $f(x) = ax^3$ and $F(x) = \frac{1}{4}ax^4$ for some constant $a \in \mathbb{R}$ and all $x \in \mathbb{R}$. Now let $k = 4$ then we get,

$$\tfrac{R_4}{L_4}\sum_{i=1}^{2}\gamma_i a_i^5 = \sum_{j=1}^{8}\alpha_j c_j^4,$$

$$\tfrac{R_4}{L_4}\sum_{i=1}^{2}\gamma_i b_i^5 = \sum_{j=1}^{8}\beta_j d_j^4,$$

and

$$\tfrac{R_4}{L_4}\sum_{i=1}^{2}\binom{5}{p}\gamma_i a_i^p b_i^{5-p} = 0 \neq \sum_{j=1}^{8}\binom{4}{p}\beta_j c_j^p d_j^{4-p} + \sum_{j=1}^{8}\binom{4}{p-1}\alpha_j c_j^{p-1} d_j^{5-p},$$

for some $p \in \{1, 2, 3, 4\}$. In particular take $p = 1$ then we see that

$$\tfrac{R_4}{L_4}\sum_{i=1}^{2}5\gamma_i a_i b_i^4 = 0 \neq -\tfrac{4}{27} = \sum_{j=1}^{8}4\beta_j c_j d_j^3 + \sum_{j=1}^{8}\alpha_j d_j^4.$$

Hence, this leads to $f = F = 0$. Finally, if $k = 5$, then we obtain

$$\frac{R_5}{L_5}\sum_{i=1}^{2}\gamma_i a_i^6 = \sum_{j=1}^{8}\alpha_j c_j^5,$$

$$\frac{R_5}{L_5}\sum_{i=1}^{2}\gamma_i b_i^6 = \sum_{j=1}^{8}\beta_j d_j^5,$$

and

$$\frac{R_5}{L_5}\sum_{i=1}^{2}\binom{6}{p}\gamma_i a_i^p b_i^{6-p} = 0 \neq \sum_{j=1}^{8}\binom{5}{p}\beta_j c_j^p d_j^{5-p} + \sum_{j=1}^{8}\binom{5}{p-1}\alpha_j c_j^{p-1} d_j^{6-p},$$

for some $p \in \{1,2,3,4,5\}$. In particular take $p = 1$ then we see that

$$\frac{R_5}{L_5}\sum_{i=1}^{2}6\gamma_i a_i b_i^5 = 0 \neq -\frac{8}{27} = \sum_{j=1}^{8}5\beta_j c_j d_j^4 + \sum_{j=1}^{8}\alpha_j d_j^5.$$

Hence, this leads to $f = F = 0$. Now taking into account Proposition 5.1.1 we see that, if $k = 0$, $L_0 = \sum_{i=1}^{2}\gamma_i(a_i + b_i) = 0$ then $f = 0$ and $F = e$, where e is a real number, is also a solution to (5.2.1), because $\sum_{i=1}^{2}\gamma_i = 0$. Thus the general solution of equation (5.2.1) is given by $f(x) = ax^3 + bx^2 + cx + d$ and $F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e$ where $x \in \mathbb{R}$ and $a, b, c, d, e \in \mathbb{R}$. To finish the proof it suffices to check that these functions satisfy equation (5.2.1). □

Theorem 5.2.2. *(cf. Theorem 2 in [19] and Theorem 2.5 in [26]) The functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$F(y) - F(x) = \frac{1}{6}yf(x) + \frac{2}{3}yf\left(\frac{x+y}{2}\right) + \frac{1}{6}yf(y) - \frac{1}{6}xf(x) - \frac{2}{3}xf\left(\frac{x+y}{2}\right) - \frac{1}{6}xf(y) \qquad (5.2.4)$$

for $x, y \in \mathbb{R}$, if and only if

$$f(x) = ax^3 + bx^2 + cx + d$$

and

$$F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e$$

for all $x \in \mathbb{R}$ and $a, b, c, d, e \in \mathbb{R}$.

Proof. Suppose that the pair (F, f) satisfies equation (5.2.4), then putting $y = x + y$ in the equation we get

$$F(x+y) - F(x) = \frac{1}{6}yf(x) + \frac{2}{3}yf\left(\frac{2x+y}{2}\right) + \frac{1}{6}yf(x+y). \qquad (5.2.5)$$

Now rearranging (5.2.5) in the form

$$\frac{1}{6}yf(x) + F(x) = F(x+y) - \frac{2}{3}yf\left(\frac{2x+y}{2}\right) - \frac{1}{6}yf(x+y), \qquad (5.2.6)$$

and applying Lemma 3.1.1 we get, $I_{0,0} = \{(id, id)\}$, $I_{0,1} = \{(id, \frac{1}{2}id), (id, id)\}$, $\psi_{0,0,(id,id)} = F$, $\psi_{0,1,(id,\frac{1}{2}id)} = -f$, $\psi_{0,1,(id,id)} = -f$, $\varphi_0 = F$, $\varphi_1 = f$. We also have $K_0 = I_{0,0}$, $K_1 = I_{0,1}$, and $K_0 \cup K_1 = \{(id, \frac{1}{2}id), (id, id)\}$. Therefore, $\varphi_1 = f$ is a polynomial function of degree at most $m = 3$ i.e.

$$m = \mathrm{card}(\mathrm{K}_0 \cup \mathrm{K}_1) + \mathrm{card}(\mathrm{K}_1) - 1 = 2 + 2 - 1 = 3.$$

Since (5.2.4) is a special case of (5.1.1) we know also that F is a polynomial function. Now we check conditions of Theorem 5.1.1. If $k = 0$, then $f(x) = d$, for some constant $d \in \mathbb{R}$ and all $x \in \mathbb{R}$, further from (5.1.20) and (5.1.21) we have

$$\frac{R_0}{L_0}\sum_{i=1}^{2}\gamma_i a_i = \sum_{j=1}^{6}\alpha_j,$$

and

$$\frac{R_0}{L_0}\sum_{i=1}^{2}\gamma_i b_i = \sum_{j=1}^{6}\beta_j.$$

Hence, $\frac{R_0}{L_0}=1$, and consequently $F(x)=dx$ for some constant $d\in\mathbb{R}$ and all $x\in\mathbb{R}$. Now let $k=1$ we get

$$\frac{R_1}{L_1}\sum_{i=1}^{2}\gamma_i a_i^2 = \sum_{j=1}^{6}\alpha_j c_j,$$

$$\frac{R_1}{L_1}\sum_{i=1}^{2}\gamma_i b_i^2 = \sum_{j=1}^{6}\beta_j d_j,$$

and

$$\frac{R_1}{L_1}\sum_{i=1}^{2}2\gamma_i a_i b_i = 0 = \sum_{j=1}^{6}\beta_j c_j + \sum_{j=1}^{6}\alpha_j d_j,$$

thus, $\frac{R_1}{L_1}=\frac{1}{2}$ and also,

$$\sum_{j=1}^{6}\beta_j c_j = \tfrac{1}{2} \neq -\tfrac{1}{2} = \sum_{j=1}^{6}\alpha_j d_j$$

then by Theorem 5.1.1 we infer that the monomial functions F, f are continuous, therefore $f(x)=cx$ and $F(x)=\frac{1}{2}cx^2$ for some constant $c\in\mathbb{R}$ and all $x\in\mathbb{R}$. If $k=2$ then we have

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i a_i^3 = \sum_{j=1}^{6}\alpha_j c_j^2,$$

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i b_i^3 = \sum_{j=1}^{6}\beta_j d_j^2,$$

and

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\binom{3}{p}\gamma_i a_i^p b_i^{3-p} = 0 = \sum_{j=1}^{6}\binom{2}{p}\beta_j c_j^p d_j^{2-p} + \sum_{j=1}^{6}\binom{2}{p-1}\alpha_j c_j^{p-1} d_j^{3-p},$$

for each $p\in\{1,2\}$. Hence, $\frac{R_2}{L_2}=\frac{1}{3}$ and also,

$$\sum_{j=1}^{6}\beta_j c_j^p d_j^{2-p} = \tfrac{1}{3} \neq -\tfrac{1}{3} = \sum_{j=1}^{6}\alpha_j c_j^{p-1} d_j^{3-p},$$

for each $p\in\{1,2\}$. By Theorem 5.1.1 we infer that the monomial functions F, f are continuous, therefore $f(x)=bx^2$ and $F(x)=\frac{1}{3}bx^3$ for some constant $b\in\mathbb{R}$ and all $x\in\mathbb{R}$. Finally, if $k=3$, then we obtain

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i a_i^4 = \sum_{j=1}^{6}\alpha_j c_j^3,$$

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i b_i^4 = \sum_{j=1}^{6}\beta_j d_j^3,$$

and

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\binom{4}{p}\gamma_i a_i^p b_i^{4-p} = 0 = \sum_{j=1}^{6}\binom{3}{p}\beta_j c_j^p d_j^{3-p} + \sum_{j=1}^{6}\binom{3}{p-1}\alpha_j c_j^{p-1} d_j^{4-p},$$

for each $p\in\{1,2,3\}$. Hence, $\frac{R_3}{L_3}=\frac{1}{4}$ and also,

$$\sum_{j=1}^{6}\beta_j c_j^p d_j^{3-p} = \tfrac{1}{4} \neq -\tfrac{1}{4} = \sum_{j=1}^{6}\alpha_j c_j^{p-1} d_j^{4-p},$$

for each $p\in\{1,2,3\}$. Again by Theorem 5.1.1 we infer that the monomial functions F, f are continuous, therefore $f(x)=ax^3$ and $F(x)=\frac{1}{4}ax^4$ for some constant $a\in\mathbb{R}$ and all $x\in\mathbb{R}$. Now taking into account Proposition 5.1.1 we see that, if $k=0$, $L_0=\sum_{i=1}^{2}\gamma_i(a_i+b_i)=0$ then

$f = 0$ and $F = e$ where e is a real number, is also a solution to (5.2.4), since $\sum_{i=1}^{2} \gamma_i = 0$. Thus the general solution of equation (5.2.4) is given by $f(x) = ax^3 + bx^2 + cx + d$ and $F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e$ where $x \in \mathbb{R}$ and $a, b, c, d, e \in \mathbb{R}$. To finish the proof it suffices to check that these functions satisfy equation (5.2.4). □

Remark 5.2.1. *(cf. Remark 2.4 in [26]) If in (5.1.1) $n = 2, \gamma_1 = 1, \gamma_2 = -1, a_1 = b_2 = 1, b_1 = a_2 = 0$, and $\beta_j = -\alpha_j$ for each $j \in \{1, \cdots, m\}$ then we get the equation considered by B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [19] namely,*

$$F(x) - F(y) = (x - y)[\alpha_1 f(c_1 x + d_1 y) + \cdots + \alpha_m f(c_m x + d_m y)].$$

It is worth noting that (5.2.1) stems from a well known quadrature rule used in numerical analysis.

5.2.2 Functional equations connected with Hermite-Hadamard quadrature rules

In line with the papers of B. Koclęga-Kulpa, T. Szostok in [18] and B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [20], we now consider polynomial functions connected with Hermite-Hadamard inequality in the class of continuous functions. The Hermite-Hadamard inequality is given as

$$f\left(\tfrac{x+y}{2}\right) \leqslant \tfrac{1}{y-x} \int_x^y f(t)\, dt \leqslant \tfrac{f(x)+f(y)}{2}, \tag{5.2.7}$$

for all $x, y \in \mathbb{R}$. Rewrite now inequality (5.2.7) in the form

$$\tfrac{1}{y-x} \int_x^y f(t)\, dt \in \left[f\left(\tfrac{x+y}{2}\right), \tfrac{f(x)+f(y)}{2}\right],$$

for all $x, y \in \mathbb{R}$. However, if we consider the function $f(x) = x^3 + x^2 + x$, then we have much more detailed information namely,

$$\tfrac{1}{y-x} \int_x^y f(t)\, dt = \tfrac{2}{3} f\left(\tfrac{x+y}{2}\right) + \tfrac{1}{3} \tfrac{f(x)+f(y)}{2}. \tag{5.2.8}$$

Now we may rewrite (5.2.8) in the form

$$F(y) - F(x) = (y - x)\left(\tfrac{2}{3} f\left(\tfrac{x+y}{2}\right) + \tfrac{1}{3} \tfrac{f(x)+f(y)}{2}\right), \tag{5.2.9}$$

where $F' = f$ (because f is continuous). Now combining equations (5.2.8) and (5.2.9) we obtain a more general functional equation namely,

$$F(y) - F(x) = \tfrac{1}{y-x} \int_x^y f(t)\, dt = (y - x) \sum_{j=1}^{m} \beta_j f(c_j x + (1 - c_j) y), \tag{5.2.10}$$

for every $x, y \in \mathbb{R}$, $c_j \in \mathbb{Q}$, and $\beta_j \in \mathbb{R}$ with $\sum_{j=1}^{m} \beta_j = 1$. This equation is related to the approximate integration. Note that the quadrature rules of an approximate integration can be obtained by the appropriate specification of the coefficients of (5.2.10).

Remark 5.2.2. *(cf. Remark 2.5 in [26]) Observe that in (5.1.1), if $n = 2, \gamma_1 = 1, \gamma_2 = -1, a_1 = b_2 = 0, b_1 = a_2 = 1$, $\alpha_j = -\beta_j$ for each $j \in \{1, \cdots, m\}$ with $\sum_{j=1}^{m} \beta_j = 1$, and $d_j = 1 - c_j$ for each $j \in \{1, \cdots, m\}$ then we obtain equation (5.2.10) which is the functional equation considered by B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [20]. We note here that in their paper $c_j \in \mathbb{R}$.*

Since (5.2.10) is a special form of (5.1.1), we may now use our method to obtain the polynomial functions of the functional equations belonging to class (5.2.10).

Theorem 5.2.3. *(cf. Theorem 2.6 in [26]) The functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$F(y) - F(x) = (y - x)\left(\tfrac{2}{3} f\left(\tfrac{x+y}{2}\right) + \tfrac{1}{3}\tfrac{f(x)+f(y)}{2}\right) \tag{5.2.11}$$

for $x, y \in \mathbb{R}$, if and only if

$$f(x) = ax^3 + bx^2 + cx + d$$

and

$$F(x) = \tfrac{1}{4}ax^4 + \tfrac{1}{3}bx^3 + \tfrac{1}{2}cx^2 + dx + e$$

for all $x \in \mathbb{R}$ and $a, b, c, d, e \in \mathbb{R}$.

Proof. Suppose that the pair (F, f) satisfies equation (5.2.11), then putting $y = x + y$ in the equation and applying Lemma 3.1.1 we get that f is a polynomial function of degree at most 3. Since (5.2.11) is a special case of (5.1.1) we know also that F is a polynomial function. Now we rewrite equation (5.2.11) in the form,

$$F(y) - F(x) = \tfrac{2}{3}yf\left(\tfrac{x+y}{2}\right) + \tfrac{1}{6}yf(x) + \tfrac{1}{6}yf(y) - \tfrac{2}{3}xf\left(\tfrac{x+y}{2}\right) - \tfrac{1}{6}xf(x) - \tfrac{1}{6}xf(y),$$

and check conditions of Theorem 5.1.1. If $k = 0$, then $f(x) = d$, for some constant $d \in \mathbb{R}$ and all $x \in \mathbb{R}$, further from (5.1.20) and (5.1.21) we have

$$\tfrac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i a_i = \sum_{j=1}^{6} \alpha_j,$$

and

$$\tfrac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i b_i = \sum_{j=1}^{6} \beta_j.$$

Hence, $\frac{R_0}{L_0} = 1$, thus, $F(x) = dx$ for some constant $d \in \mathbb{R}$ and all $x \in \mathbb{R}$. If $k = 1$ then we get

$$\tfrac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i a_i^2 = \sum_{j=1}^{6} \alpha_j c_j,$$

$$\tfrac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i b_i^2 = \sum_{j=1}^{6} \beta_j d_j,$$

and

$$\tfrac{R_1}{L_1} \sum_{i=1}^{2} 2\gamma_i a_i b_i = 0 = \sum_{j=1}^{6} \beta_j c_j + \sum_{j=1}^{6} \alpha_j d_j$$

thus, $\frac{R_1}{L_1} = \frac{1}{2}$ and also,

$$\sum_{j=1}^{6} \beta_j c_j = \tfrac{1}{2} \neq -\tfrac{1}{2} = \sum_{j=1}^{6} \alpha_j d_j$$

thus by Theorem 5.1.1 we have that the monomial functions F, f are continuous, therefore $f(x) = cx$ and $F(x) = \frac{1}{2}cx^2$ for some constant $c \in \mathbb{R}$ and all $x \in \mathbb{R}$. Now let $k = 2$ then we have

$$\tfrac{R_2}{L_2} \sum_{i=1}^{2} \gamma_i a_i^3 = \sum_{j=1}^{6} \alpha_j c_j^2,$$

$$\tfrac{R_2}{L_2} \sum_{i=1}^{2} \gamma_i b_i^3 = \sum_{j=1}^{6} \beta_j d_j^2,$$

and

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\binom{3}{p}\gamma_i a_i^p b_i^{3-p} = 0 = \sum_{j=1}^{6}\binom{2}{p}\beta_j c_j^p d_j^{2-p} + \sum_{j=1}^{6}\binom{2}{p-1}\alpha_j c_j^{p-1} d_j^{3-p},$$

for each $p \in \{1,2\}$. Hence, $\frac{R_2}{L_2} = \frac{1}{3}$ and also,

$$\sum_{j=1}^{6}\beta_j c_j^p d_j^{2-p} = \tfrac{1}{3} \neq -\tfrac{1}{3} = \sum_{j=1}^{6}\alpha_j c_j^{p-1} d_j^{3-p},$$

for each $p \in \{1,2\}$. By Theorem 5.1.1 we have that the monomial functions F, f are continuous, therefore $f(x) = bx^2$ and $F(x) = \frac{1}{3}bx^3$ for some constant $b \in \mathbb{R}$ and all $x \in \mathbb{R}$. Finally, if $k = 3$, then we obtain

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i a_i^4 = \sum_{j=1}^{6}\alpha_j c_j^3,$$

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i b_i^4 = \sum_{j=1}^{6}\beta_j d_j^3,$$

and

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\binom{4}{p}\gamma_i a_i^p b_i^{4-p} = 0 = \sum_{j=1}^{6}\binom{3}{p}\beta_j c_j^p d_j^{3-p} + \sum_{j=1}^{6}\binom{3}{p-1}\alpha_j c_j^{p-1} d_j^{4-p},$$

for each $p \in \{1,2,3\}$. Hence, $\frac{R_3}{L_3} = \frac{1}{4}$ and also,

$$\sum_{j=1}^{6}\beta_j c_j^p d_j^{3-p} = \tfrac{1}{4} \neq -\tfrac{1}{4} = \sum_{j=1}^{6}\alpha_j c_j^{p-1} d_j^{4-p},$$

for each $p \in \{1,2,3\}$. Again by Theorem 5.1.1 we infer that the monomial functions F, f are continuous, therefore $f(x) = ax^3$ and $F(x) = \frac{1}{4}ax^4$ for some constant $a \in \mathbb{R}$ and all $x \in \mathbb{R}$. Now by Proposition 5.1.1, we see that if $k = 0$, $L_0 = \sum_{i=1}^{2}\gamma_i(a_i + b_i) = 0$ then $f = 0$ and $F = e$ where $e \in \mathbb{R}$ is also a solution to (5.2.11), since $\sum_{i=1}^{2}\gamma_i = 0$. Thus the general solution of equation (5.2.11) is given by $f(x) = ax^3 + bx^2 + cx + d$ and $F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e$ where $x \in \mathbb{R}$ and $a, b, c, d, e \in \mathbb{R}$. To finish the proof it suffices to check that these functions satisfy equation (5.2.11). □

Theorem 5.2.4. *(cf. Theorem 4 in [20] and Theorem 2.7 in [26]) The functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$F(y) - F(x) = (y - x)\left(\tfrac{1}{4}f(x) + \tfrac{3}{4}f\left(\tfrac{1}{3}x + \tfrac{2}{3}y\right)\right) \tag{5.2.12}$$

for $x, y \in \mathbb{R}$, if and only if

$$f(x) = ax^2 + bx + c$$

and

$$F(x) = \tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx + d$$

for all $x \in \mathbb{R}$ and $a, b, c, d \in \mathbb{R}$.

Proof. Suppose that the pair (F, f) satisfies equation (5.2.12), then substituting $y = x + y$ in the equation and applying Lemma 3.1.1 we get that f is a polynomial function of degree at most 2. Since (5.2.12) is a special case of (5.1.1) we have that F is a polynomial function. Now we rewrite equation (5.2.12) in the form,

$$F(y) - F(x) = \tfrac{1}{4}yf(x) + \tfrac{3}{4}yf\left(\tfrac{1}{3}x + \tfrac{2}{3}y\right) - \tfrac{1}{4}xf(x) - \tfrac{3}{4}xf\left(\tfrac{1}{3}x + \tfrac{2}{3}y\right),$$

and check conditions of Theorem 5.1.1. If $k=0$, then $f(x)=c$, for some constant $c\in\mathbb{R}$ and all $x\in\mathbb{R}$, further from (5.1.20) and (5.1.21) we have

$$\frac{R_0}{L_0}\sum_{i=1}^{2}\gamma_i a_i=\sum_{j=1}^{4}\alpha_j,$$

and

$$\frac{R_0}{L_0}\sum_{i=1}^{2}\gamma_i b_i=\sum_{j=1}^{4}\beta_j.$$

Hence, $\frac{R_0}{L_0}=1$, thus, $F(x)=cx$ for some constant $c\in\mathbb{R}$ and all $x\in\mathbb{R}$. If $k=1$ then we get

$$\frac{R_1}{L_1}\sum_{i=1}^{2}\gamma_i a_i^2=\sum_{j=1}^{4}\alpha_j c_j,$$

$$\frac{R_1}{L_1}\sum_{i=1}^{2}\gamma_i b_i^2=\sum_{j=1}^{4}\beta_j d_j,$$

and

$$\frac{R_1}{L_1}\sum_{i=1}^{2}2\gamma_i a_i b_i=0=\sum_{j=1}^{4}\beta_j c_j+\sum_{j=1}^{4}\alpha_j d_j,$$

thus, $\frac{R_1}{L_1}=\frac{1}{2}$ and also,

$$\sum_{j=1}^{4}\beta_j c_j=\frac{1}{2}\neq-\frac{1}{2}=\sum_{j=1}^{4}\alpha_j d_j,$$

therefore by Theorem 5.1.1 we have that the monomial functions F,f are continuous, hence $f(x)=bx$ and $F(x)=\frac{1}{2}bx^2$ for some constant $b\in\mathbb{R}$ and all $x\in\mathbb{R}$. Finally, if $k=2$ then we have

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i a_i^3=\sum_{j=1}^{4}\alpha_j c_j^2,$$

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i b_i^3=\sum_{j=1}^{4}\beta_j d_j^2,$$

and

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\binom{3}{p}\gamma_i a_i^p b_i^{3-p}=0=\sum_{j=1}^{4}\binom{2}{p}\beta_j c_j^p d_j^{2-p}+\sum_{j=1}^{4}\binom{2}{p-1}\alpha_j c_j^{p-1}d_j^{3-p},$$

for each $p\in\{1,2\}$. Hence, $\frac{R_2}{L_2}=\frac{1}{3}$ and also,

$$\sum_{j=1}^{4}\beta_j c_j^p d_j^{2-p}=\frac{1}{3}\neq-\frac{1}{3}=\sum_{j=1}^{4}\alpha_j c_j^{p-1}d_j^{3-p},$$

for each $p\in\{1,2\}$. By Theorem 5.1.1 we have that the monomial functions (F,f) are continuous, therefore $f(x)=ax^2$ and $F(x)=\frac{1}{3}ax^3$ for some constant $a\in\mathbb{R}$ and all $x\in\mathbb{R}$. Now by Proposition 5.1.1, we see that if $k=0$, $L_0=\sum_{i=1}^{2}\gamma_i(a_i+b_i)=0$ then $f=0$ and $F=d$ where $d\in\mathbb{R}$ is also a solution to (5.2.12), because $\sum_{i=1}^{2}\gamma_i=0$. Therefore, the general solution of equation (5.2.12) is given by $f(x)=ax^2+bx+c$ and $F(x)=\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx+d$ where $x\in\mathbb{R}$ and $a,b,c,d\in\mathbb{R}$. To finish the proof it suffices to check that these functions satisfy equation (5.2.12). □

Now we give some examples that include known results which may be solved by the use of our method.

Example 5.2.1. *(cf. Example 2.1 in [26]) Assume that the functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$F(x) = yf(x) \tag{5.2.13}$$

for all $x, y \in \mathbb{R}$.

Now we rearrange (5.2.13) in the form

$$yf(x) - F(x) = 0, \tag{5.2.14}$$

for all $x, y \in \mathbb{R}$. Applying Lemma 3.1.1, we can see that f is the zero function. Clearly, (5.2.13) is a special case of (5.1.1), thus we infer that F is also a polynomial function. Now, check conditions of Theorem 5.1.1 and taking into account Remark 5.1.3, we see that $f = F = 0$ is the only solution of (5.2.13).

5.2.3 Functional equations connected with Lagrange mean value theorem

Example 5.2.2. *(J. Aczél result cf. [1] and Example 2.2 in [26]) The functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\tfrac{F(y)-F(x)}{y-x} = f(x+y) \tag{5.2.15}$$

for all $x, y \in \mathbb{R}$, if and only if

$$f(x) = ax + b$$

and

$$F(x) = ax^2 + bx + c$$

for all $x, y \in \mathbb{R}$, and $a, b, c \in \mathbb{R}$

Proof. Now we rewrite (5.2.15) in the form of equation (5.1.1)

$$F(y) - F(x) = yf(x+y) - xf(x+y). \tag{5.2.16}$$

Suppose that the pair (F, f) satisfies (5.2.16), then rearranging (5.2.16) in the form

$$F(x) = F(y) - yf(x+y) + xf(x+y), \tag{5.2.17}$$

and applying Lemma 3.1.1, we get $I_{0,0} = \{(0, id)\}$, $I_{0,1} = I_{1,0} = \{(id, id)\}$, $\psi_{0,0,(0,id)} = F$, $\psi_{0,1,(id,id)} = -f$, $\psi_{1,0,(id,id)} = f$, $\varphi_0 = F$. We also have $K_0 = I_{0,0}$, $K_1 = I_{0,1} \cup I_{1,0}$, and $K_0 \cup K_1 = \{(0, id), (id, id)\}$. Therefore, $\varphi_0 = F$ is a polynomial function of degree at most $m = 2$ i.e.

$$m = \mathrm{card}(\mathrm{K}_0 \cup \mathrm{K}_1) + \mathrm{card}(\mathrm{K}_1) - 1 = 2 + 1 - 1 = 2.$$

Since (5.2.16) is a special form of (5.1.1), thus we know also that f is a polynomial function. By Theorem 5.1.1 we infer that f is at most degree 1. Now we check conditions of Theorem 5.1.1. If $k = 0$, then $f(x) = b$, for some constant $b \in \mathbb{R}$ and all $x \in \mathbb{R}$, further from (5.1.20) and (5.1.21) we have

$$\tfrac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i a_i = \sum_{j=1}^{2} \alpha_j,$$

and

$$\tfrac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i b_i = \sum_{j=1}^{2} \beta_j.$$

Hence, $\frac{R_0}{L_0} = 1$, and consequently $F(x) = bx$ for some constant $b \in \mathbb{R}$ and all $x \in \mathbb{R}$. Now let $k = 1$ we get

$$\frac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i a_i^2 = \sum_{j=1}^{2} \alpha_j c_j,$$

$$\frac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i b_i^2 = \sum_{j=1}^{2} \beta_j d_j,$$

and

$$\frac{R_1}{L_1} \sum_{i=1}^{2} 2\gamma_i a_i b_i = 0 = \sum_{j=1}^{2} \beta_j c_j + \sum_{j=1}^{2} \alpha_j d_j,$$

thus, $\frac{R_1}{L_1} = 1$ and also,

$$\sum_{j=1}^{2} \beta_j c_j = 1 \neq -1 = \sum_{j=1}^{2} \alpha_j d_j,$$

hence by Theorem 5.1.1 we infer that the monomial functions F, f are continuous, therefore $f(x) = ax$ and $F(x) = ax^2$ for some constant $a \in \mathbb{R}$ and all $x \in \mathbb{R}$. Now by Proposition 5.1.1, we see that if $k = 0$, $L_0 = \sum_{i=1}^{2} \gamma_i(a_i + b_i) = 0$ then $f = 0$ and $F = c$ where $c \in \mathbb{R}$ is also a solution to (5.2.15), because $\sum_{i=1}^{2} \gamma_i = 0$. Thus the general solution of equation (5.2.15) is given by $f(x) = ax + b$ and $F(x) = ax^2 + bx + c$ where $x \in \mathbb{R}$ and $a, b, c \in \mathbb{R}$. To finish the proof it suffices to check that these functions satisfy equation (5.2.15). □

Remark 5.2.3. *(cf. Remark 2.6 in [26]) We note here that the functional equation considered by J. Aczél in [1] is a special case of equation (5.1.1). In particular, choose* $n = 2, m = 2, \gamma_1 = \beta_1 = 1, \gamma_2 = \alpha_2 = -1, \alpha_1 = \beta_2 = 0, a_1 = b_2 = 0$ *and* $b_1 = a_2 = c_1 = d_1 = c_2 = d_2 = 1$, *we get*

$$F(y) - F(x) = (y - x)f(x + y)$$

Example 5.2.3. *(cf. Theorem 5 in [2] and Example 2.3 in [26]) The functions* $F, f : \mathbb{R} \to \mathbb{R}$ *satisfy*

$$\frac{F(x)-F(y)}{x-y} = f\left(\frac{x+y}{2}\right) \tag{5.2.18}$$

for all $x, y \in \mathbb{R}$, *if and only if*

$$f(x) = ax + b$$

and

$$F(x) = \tfrac{1}{2}ax^2 + bx + c$$

for all $x, y \in \mathbb{R}$, *and* $a, b, c \in \mathbb{R}$.

Proof. Now we rewrite (5.2.18) in the form of equation (5.1.1)

$$F(x) - F(y) = xf\left(\frac{x+y}{2}\right) - yf\left(\frac{x+y}{2}\right). \tag{5.2.19}$$

Suppose that the pair (F, f) satisfies (5.2.19), then rearranging (5.2.19) in the form

$$F(x) = F(y) + xf\left(\frac{x+y}{2}\right) - yf\left(\frac{x+y}{2}\right), \tag{5.2.20}$$

and applying Lemma 3.1.1 we get $I_{0,0} = \{(0, id)\}$, $I_{1,0} = I_{0,1} = \{(\frac{1}{2}id, \frac{1}{2}id)\}$, $\psi_{0,0,(0,id)} = F$, $\psi_{1,0,(\frac{1}{2}id,\frac{1}{2}id)} = f$, $\psi_{0,1,(\frac{1}{2}id,\frac{1}{2}id)} = -f$, $\varphi_0 = F$. We also have $K_0 = I_{0,0}$, $K_1 = I_{1,0} \cup I_{0,1}$, and $K_0 \cup K_1 = \{(0, id), (\frac{1}{2}id, \frac{1}{2}id)\}$. Therefore, $\varphi_0 = F$ is a polynomial function of degree at most $m = 2$ i.e.

$$m = \mathrm{card}(\mathrm{K}_0 \cup \mathrm{K}_1) + \mathrm{card}(\mathrm{K}_1) - 1 = 2 + 1 - 1 = 2.$$

Clearly, (5.2.19) is a special form of (5.1.1), thus we know also that f is a polynomial function. By Theorem 5.1.1 we infer that f is at most degree 1. Now we check conditions of Theorem 5.1.1. If $k = 0$, then $f(x) = b$, for some constant $b \in \mathbb{R}$ and all $x \in \mathbb{R}$, further from (5.1.20) and (5.1.21) we have

$$\frac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i a_i = \sum_{j=1}^{2} \alpha_j,$$

and

$$\frac{R_0}{L_0} \sum_{i=1}^{2} \gamma_i b_i = \sum_{j=1}^{2} \beta_j.$$

Hence, $\frac{R_0}{L_0} = 1$, and consequently $F(x) = bx$ for some constant $b \in \mathbb{R}$ and all $x \in \mathbb{R}$. Finally, let $k = 1$ we get

$$\frac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i a_i^2 = \sum_{j=1}^{2} \alpha_j c_j,$$

$$\frac{R_1}{L_1} \sum_{i=1}^{2} \gamma_i b_i^2 = \sum_{j=1}^{2} \beta_j d_j,$$

and

$$\frac{R_1}{L_1} \sum_{i=1}^{2} 2\gamma_i a_i b_i = 0 = \sum_{j=1}^{2} \beta_j c_j + \sum_{j=1}^{2} \alpha_j d_j,$$

thus, $\frac{R_1}{L_1} = \frac{1}{2}$ and also,

$$\sum_{j=1}^{2} \beta_j c_j = -\tfrac{1}{2} \neq \tfrac{1}{2} = \sum_{j=1}^{2} \alpha_j d_j$$

then by Theorem 5.1.1 we infer that the monomial functions F, f are continuous, therefore $f(x) = ax$ and $F(x) = \frac{1}{2}ax^2$ for some constant $a \in \mathbb{R}$ and all $x \in \mathbb{R}$. Now by Proposition 5.1.1, we see that if $k = 0$, $L_0 = \sum_{i=1}^{2} \gamma_i(a_i + b_i) = 0$ then $f = 0$ and $F = c$ where $c \in \mathbb{R}$ is also a solution to (5.2.18), because $\sum_{i=1}^{2} \gamma_i = 0$. Thus the general solution of equation (5.2.18) is given by $f(x) = ax + b$ and $F(x) = \frac{1}{2}ax^2 + bx + c$ where $x \in \mathbb{R}$ and $a, b, c \in \mathbb{R}$. To finish the proof it suffices to check that these functions satisfy equation (5.2.18). □

Remark 5.2.4. *(cf. Remark 2.7 in [26]) We note here that using our method in solving (5.2.18) we obtained the same results as J. Aczél, M. Kuczma in [2] (cf. Theorem 5 in [2]).*

5.2.4 Functional equations connected with descriptive geometry

Example 5.2.4. *(cf. Example 2.4 in [26]) The functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy*

$$2F(y) - 2F(x) = (y - x)\left(f\left(\frac{x+y}{2}\right) + \frac{f(x)+f(y)}{2}\right) \qquad (5.2.21)$$

for $x, y \in \mathbb{R}$, if and only if

$$f(x) = ax + b$$

and

$$F(x) = \tfrac{1}{2}ax^2 + bx + c$$

for all $x \in \mathbb{R}$ and $a, b, c \in \mathbb{R}$.

Proof. Suppose that the pair (F, f) satisfies equation (5.2.21), then substituting $y = x + y$ in the equation and applying Lemma 3.1.1 we get that f is a polynomial function of degree at most 3. Since (5.2.21) is a special case of (5.1.1) we have that F is a polynomial function. Now we rewrite equation (5.2.21) in the form,

$$2F(y)-2F(x)=yf\left(\frac{x+y}{2}\right)+\frac{1}{2}yf(x)+\frac{1}{2}yf(y)-xf\left(\frac{x+y}{2}\right)-\frac{1}{2}xf(x)-\frac{1}{2}xf(y),$$

and check conditions of Theorem 5.1.1. If $k=0$, then $f(x)=b$, for some constant $b\in\mathbb{R}$ and all $x\in\mathbb{R}$, further from (5.1.20) and (5.1.21) we have

$$\frac{R_0}{L_0}\sum_{i=1}^{2}\gamma_i a_i=\sum_{j=1}^{6}\alpha_j,$$

and

$$\frac{R_0}{L_0}\sum_{i=1}^{2}\gamma_i b_i=\sum_{j=1}^{6}\beta_j.$$

Hence, $\frac{R_0}{L_0}=1$, thus, $F(x)=bx$ for some constant $b\in\mathbb{R}$ and all $x\in\mathbb{R}$. If $k=1$ then we get

$$\frac{R_1}{L_1}\sum_{i=1}^{2}\gamma_i a_i^2=\sum_{j=1}^{6}\alpha_j c_j,$$

$$\frac{R_1}{L_1}\sum_{i=1}^{2}\gamma_i b_i^2=\sum_{j=1}^{6}\beta_j d_j,$$

and

$$\frac{R_1}{L_1}\sum_{i=1}^{2}2\gamma_i a_i b_i=0=\sum_{j=1}^{6}\beta_j c_j+\sum_{j=1}^{6}\alpha_j d_j,$$

thus, $\frac{R_1}{L_1}=\frac{1}{2}$ and also,

$$\sum_{j=1}^{6}\beta_j c_j=\frac{3}{2}\neq-\frac{3}{2}=\sum_{j=1}^{6}\alpha_j d_j,$$

hence by Theorem 5.1.1 we have that the monomial functions F,f are continuous, therefore $f(x)=ax$ and $F(x)=\frac{1}{2}ax^2$ for some constant $a\in\mathbb{R}$ and all $x\in\mathbb{R}$. Now let $k=2$ then we get

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i a_i^3=\sum_{j=1}^{6}\alpha_j c_j^2,$$

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\gamma_i b_i^3=\sum_{j=1}^{6}\beta_j d_j^2,$$

and

$$\frac{R_2}{L_2}\sum_{i=1}^{2}\binom{3}{p}\gamma_i a_i^p b_i^{3-p}=0\neq\sum_{j=1}^{6}\binom{2}{p}\beta_j c_j^p d_j^{2-p}+\sum_{j=1}^{6}\binom{2}{p-1}\alpha_j c_j^{p-1}d_j^{3-p},$$

for some $p\in\{1,2\}$. In particular take $p=1$ then we see that

$$\frac{R_2}{L_2}\sum_{i=1}^{2}3\gamma_i a_i b_i^2=0\neq-\frac{1}{4}=\sum_{j=1}^{6}2\beta_j c_j d_j+\sum_{j=1}^{6}\alpha_j d_j^2.$$

Hence, this leads to $f=F=0$. Finally, if $k=3$, then we obtain

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i a_i^4=\sum_{j=1}^{6}\alpha_j c_j^3,$$

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\gamma_i b_i^4=\sum_{j=1}^{6}\beta_j d_j^3,$$

and

$$\frac{R_3}{L_3}\sum_{i=1}^{2}\binom{4}{p}\gamma_i a_i^p b_i^{4-p}=0\neq\sum_{j=1}^{6}\binom{3}{p}\beta_j c_j^p d_j^{3-p}+\sum_{j=1}^{6}\binom{3}{p-1}\alpha_j c_j^{p-1}d_j^{4-p},$$

for some $p \in \{1,2,3\}$. In particular take $p = 1$ then we see that

$$\frac{R_3}{L_3}\sum_{i=1}^{2} 4\gamma_i a_i b_i^3 = 0 \neq -\frac{1}{4} = \sum_{j=1}^{6} 3\beta_j c_j d_j^2 + \sum_{j=1}^{6} \alpha_j d_j^3.$$

Again by Theorem 5.1.1, this leads to $f = F = 0$. Now by Proposition 5.1.1, we see that if $k = 0$, $L_0 = \sum_{i=1}^{2} \gamma_i(a_i + b_i) = 0$ then $f = 0$ and $F = c$ where $c \in \mathbb{R}$ is also a solution to (5.2.21), because $\sum_{i=1}^{2} \gamma_i = 0$. Thus the general solution of equation (5.2.21) is given by $f(x) = ax + b$ and $F(x) = \frac{1}{2}ax^2 + bx + c$ where $x \in \mathbb{R}$ and $a, b, c \in \mathbb{R}$. To finish the proof it suffices to check that these functions satisfy equation (5.2.21). □

Remark 5.2.5. *(cf. Remark 2.8 in [26]) We note here that (5.2.21) is the functional equation arising from the geometric problems considered by C. Alsina, M. Sablik and J. Sikorska in [4].*

Remark 5.2.6. *(cf. Remark 2.9 in [26]) Let us observe that when $n = 3, \gamma_1 = a_1 = b_1 = a_2 = b_3 = 1, a_3 = b_2 = 0,\ \gamma_2 = \gamma_3 = -1,\ m = 2,\ \alpha_1 = \beta_2 = d_1 = c_2 = 1,$ and $\alpha_2 = \beta_1 = c_1 = d_2 = 0$, equations (5.1.1), (5.1.2) and (5.1.3) have the same polynomial solutions (see Chapter 3(cf.[25]) and Chapter 4 (cf. [28])) as equation (5.1.9) considered by W. Fechner and E. Gselmann in [11]. In addition, the polynomial solutions of equations (5.1.2) or (5.1.3) are also polynomial solutions of equation (5.1.1) but the converse is not necessarily true.*

Remark 5.2.7. *(cf. Remark 2.10 in [26]) To this end, we conclude that the main results of Chapter 3 (see Theorem 3.2.2 in Chapter 3 and Theorem 3.3 in [25]) and Chapter 4 (see Theorem 4.1.2 in Chapter 4 and Theorem 2.2 in [28]) are special forms of our results. Moreso, we mention here that the pair of functions (F, f) mapping from $\mathbb{R}$ to $\mathbb{R}$ that satisfies equations (5.1.1), (5.1.2) and (5.1.3) respectively, were obtained by assuming that $x = y \in \mathbb{R}$. From Chapter 4(cf. [28]), we see that it is possible to use a computer program to solve functional equations in particular, equation (5.1.3). Therefore, these leads to the following questions:*

a) Which are the polynomial functions F, f mapping $\mathbb{R}$ to $\mathbb{R}$ that satisfy equations (5.1.1), (5.1.2), (5.1.3) and (5.1.9) when $x \neq y$?

b) Is it possible to formulate a robust computer algorithm which determines the polynomial solutions of equation (5.1.1) and the polynomial solutions of question a)?

Chapter 6

Characterizing locally polynomial functions on convex subsets of linear spaces

Several mathematicians attempted to solve equations characterizing the polynomial functions of restricted domains, see e.g. J. Ger ([13]) or Z. Daróczy and Gy. Maksa ([9]). Here we introduce the notion of locally polynomial functions, satisfying a conditional Fréchet equation on convex subsets of linear space. We prove that locally polynomial functions are solutions to some rather general functional equations. Using Roman Ger's result, we infer that locally they are the restrictions of polynomial functions defined on the whole space.

We begin by proving a Lemma that generalizes the results of I. Pawlikowska [30](Theorem 2.1).

6.1 Sablik-Okeke Lemma

Lemma 6.1.1. *(cf. Lemma 1.3 in [29]) Let X and Y be two linear spaces over a field $\mathbb{K} \subset \mathbb{R}$, and let $\mathcal{K} \subset X$ be an absolutely convex set with $0 \in algint\mathcal{K}$. Fix $N \in \mathbb{N} \cup \{0\}$ and $M \in \{-1, 0, 1, \cdots\} = \mathbb{N} \cup \{-1, 0\}$. Further, if $M \geqslant 0$, assume that for each $m \in \{0, \cdots, M\}$ and each $p \in \{0, \cdots, m\}$, the set $I_{p,m-p} = \{(\alpha, \beta) \in \mathbb{K} \times \mathbb{K} : |\alpha| + |\beta| \leqslant 1,\ \beta \neq 0\}$ is finite. If the functions $\varphi_i : \mathcal{K} \to SA^i(X;Y)$, $i \in \{0, \cdots, N\}$, and, if $M \geqslant 0$,*
$\psi_{p,m-p,(\alpha,\beta)} : \mathcal{K} \to SA^m(X;Y)$, $(\alpha, \beta) \in I_{p,m-p}, m \in \{0, \cdots, M\}, p \in \{0, \cdots, m\}$ satisfy the equation

$$\sum_{i=0}^{N} \varphi_i(x)(y^i) = \begin{cases} 0, & M = -1, \\ \sum_{m=0}^{M} \sum_{p=0}^{m} \sum_{(\alpha,\beta) \in I_{p,m-p}} \psi_{p,m-p,(\alpha,\beta)} \left(\alpha x + \beta y\right) \left(x^p, y^{m-p}\right), & M \geqslant 0 \end{cases} \tag{E_M}$$

for every $x, y \in \mathcal{K}$, then there exists a $p' \in \mathbb{N}$, such that φ_N is a locally polynomial function of order at most equal

$$\sum_{m=0}^{M} \operatorname{card}\left(\bigcup_{s=m}^{M} K_s\right) - 1,$$

on the set $\frac{1}{p'}\mathcal{K}$, where $K_s = \bigcup_{p=0}^{s} I_{p,s-p}$ for each $s \in \{0, \cdots, M\}$, if $M \geqslant 0$. Moreover, if $M = -1$,

$$\sum_{i=0}^{N} \varphi_i(x)(y^i) = 0$$

then φ_N is the zero function.

Proof. Let us fix an $N \in \mathbb{N} \cup \{0\}$. We prove the lemma using induction with respect to M.

1. Let us start with $M = -1$, therefore we consider the equation

$$\sum_{i=0}^{N} \varphi_i(x)(y^i) = 0, \tag{E-1}$$

   because if $M = -1$ then the right-hand side is identically 0. Now applying Lemma 2.2.2 we infer that $B_i(y) = \varphi_i(x)(y^i) = 0$, for every $y \in \mathcal{K}$. Hence $\varphi_i(x) = 0$, $i \in \{0, \cdots, N\}$, for every $x \in \mathcal{K}$. In particular φ_N is a polynomial function identically equal to 0 and our assertion holds (φ_N is a local polynomial of degree $\leqslant 0$).

2. Assume now that Lemma 6.1.1 holds for some $M > -1$, that is, if ($\mathrm{E_M}$) is satisfied for all $x, y \in \mathcal{K}$ then there exists a $\tilde{p} \in \mathbb{N}$, such that φ_N is locally polynomial function of order at most equal to
   $\sum_{m=0}^{M} \operatorname{card}(\bigcup_{s=m}^{M} K_s) - 1$, on the set $\frac{1}{\tilde{p}}\mathcal{K}$.

3. Suppose now that (E_{M+1}) holds on $\mathcal{K}$. We are going to show that there is a $\hat{p} \in \mathbb{N}$, such that $\varphi_N|_{\frac{1}{\hat{p}}\mathcal{K}}$ is locally polynomial. In fact we are going to show that for some $\ell \in \mathbb{N}$ and every m, $\Delta_m^{\ell}\varphi_N$ is locally polynomial on $\frac{1}{\tilde{p}r}\mathcal{K}$, where $r \geqslant \max\left\{\ell+1, \max\left\{|\alpha|+|\beta|+\sum_{i=1}^{\ell}\left|\frac{\alpha\beta_i-\alpha_i\beta}{\beta_i}\right|\right\} : (\alpha,\beta) \in J\right\}$, and $\tilde{p}$ is as in 2. Here $J = \bigcup_{s=0}^{M+1} K_s = \bigcup_{s=0}^{M+1}\bigcup_{p=0}^{s} I_{p,s-p}$, and ℓ above is the cardinality of $\bigcup_{p=0}^{M+1} I_{p,M+1-p}$ which is greater or equal to 1 (because $\bigcup_{p=0}^{M+1} I_{p,M+1-p} \neq \emptyset$) but finite. We may write $\bigcup_{p=0}^{M+1} I_{p,M+1-p} = \{(\alpha_1,\beta_1), \cdots, (\alpha_\ell,\beta_\ell)\}$.

4. Let us choose a $u \in \frac{1}{r}\mathcal{K}$ and apply the operator $\Gamma_{(u,v)} : \frac{1}{r}\mathcal{K} \times \frac{1}{r}\mathcal{K} \to Y$ to both sides of (E_{M+1}), where $\Gamma_{(u,v)}\Phi(x,y) = \Phi(x+u, y+v) - \Phi(x,y)$, $u, v, x, y \in \frac{1}{r}\mathcal{K}$. Let us put $v := \frac{-\alpha_1}{\beta_1}u$. Consider the left-hand side of (E_{M+1}), we get

$$
\begin{aligned}
&\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\varphi_N(x)(y^N)+\sum_{i=0}^{N-1}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\varphi_i(x)(y^i)\\
&=\varphi_N(x+u)\left(\left(y-\frac{\alpha_1}{\beta_1}u\right)^N\right)-\varphi_N(x)(y^N)+\sum_{i=0}^{N-1}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\varphi_i(x)(y^i)\\
&=\sum_{k=0}^{N}\binom{N}{k}\varphi_N(x+u)\left(y^{N-k},\left(\tfrac{-\alpha_1}{\beta_1}u\right)^k\right)-\varphi_N(x)(y^N)\\
&+\sum_{i=0}^{N-1}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\varphi_i(x)(y^i)=\Delta_u\varphi_N(x)(y^N)\\
&+\sum_{k=1}^{N}\binom{N}{k}\varphi_N(x+u)\left(y^{N-k},\left(\tfrac{-\alpha_1}{\beta_1}u\right)^k\right)+\sum_{i=0}^{N-1}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\varphi_i(x)(y^i)\\
&=\Delta_u\varphi_N(x)(y^N)+\sum_{k=1}^{N}\binom{N}{k}\Delta_u\varphi_N(x)\left(y^{N-k},\left(\tfrac{-\alpha_1}{\beta_1}u\right)^k\right)\\
&+\sum_{k=1}^{N}\binom{N}{k}\varphi_N(x)\left(y^{N-k},\left(\tfrac{-\alpha_1}{\beta_1}u\right)^k\right)+\sum_{i=0}^{N-1}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\varphi_i(x)(y^i)\\
&=\hat\varphi_N(x)(y^N)+\sum_{i=0}^{N-1}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\varphi_i(x)(y^i)\\
&+\sum_{k=1}^{N}\binom{N}{k}\left(\hat\varphi_N(x)+\varphi_N(x)\right)\left(y^{N-k},\left(\tfrac{-\alpha_1}{\beta_1}u\right)^k\right),
\end{aligned}
$$

where $\hat\varphi_N=\Delta_u\varphi_N$. Observe that applying the Γ operator to the left-hand side of (E_{M+1}) we get the left-hand side of (E_M) but with $\hat\varphi_N$ instead of φ_N and additional summands which is a polynomial in y but of degree lower than N. Therefore, we deduce that applying the Γ operator to the left-hand side of (E_{M+1}) does not change the degree of the polynomial in y.

5. Let us see what happens with the right-hand side of (E_{M+1}), we have

$$
\begin{aligned}
(*)\quad&\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\sum_{m=0}^{M+1}\sum_{p=0}^{m}\sum_{(\alpha,\beta)\in I_{p,m-p}}\psi_{p,m-p,(\alpha,\beta)}(\alpha x+\beta y)(x^p,y^{m-p})\right]\\
&=\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\sum_{m=0}^{M}\sum_{p=0}^{m}\sum_{(\alpha,\beta)\in I_{p,m-p}}\psi_{p,m-p,(\alpha,\beta)}(\alpha x+\beta y)(x^p,y^{m-p})\right]\\
&+\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\sum_{p=0}^{M+1}\sum_{(\alpha,\beta)\in I_{p,M+1-p}}\psi_{p,M+1-p,(\alpha,\beta)}(\alpha x+\beta y)(x^p,y^{M+1-p})\right]\\
&=\sum_{m=0}^{M}\sum_{p=0}^{m}\sum_{(\alpha,\beta)\in I_{p,m-p}}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\psi_{p,m-p,(\alpha,\beta)}(\alpha x+\beta y)(x^p,y^{m-p})\right]\\
&+\sum_{p=0}^{M+1}\sum_{(\alpha,\beta)\in I_{p,M+1-p}}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\psi_{p,M+1-p,(\alpha,\beta)}(\alpha x+\beta y)(x^p,y^{M+1-p})\right]\\
&=\sum_{m=0}^{M}\sum_{p=0}^{m}\sum_{(\alpha,\beta)\in I_{p,m-p}}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\psi_{p,m-p,(\alpha,\beta)}(\alpha x+\beta y)(x^p,y^{m-p})\right]\\
&+\sum_{p=0}^{M+1}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\psi_{p,M+1-p,(\alpha_1,\beta_1)}(\alpha_1 x+\beta_1 y)(x^p,y^{M+1-p})\right]\\
&+\sum_{p=0}^{M+1}\sum_{\substack{(\alpha,\beta)\in I_{p,M+1-p}\\(\alpha,\beta)\neq(\alpha_1,\beta_1)}}\Gamma_{\left(u,\frac{-\alpha_1}{\beta_1}u\right)}\left[\psi_{p,M+1-p,(\alpha,\beta)}(\alpha x+\beta y)(x^p,y^{M+1-p})\right].
\end{aligned}
$$

The first summand is a polynomial function of degree $\leqslant M$. Now let consider the second summand. Let us note that for every $p \in \{0, \ldots, M+1\}$ the action of $\Gamma_{\left(u, \frac{-\alpha_1}{\beta_1} u\right)}$ leads to reduction of terms

$$\psi_{p,M+1-p,(\alpha_1,\beta_1)}\,(\alpha_1 x + \beta_1 y)\,(x^p, y^{M+1-p}),$$

and what is left is a polynomial of degree $\leqslant M$.

Finally, passing to the last term in $(*)$, let us fix a $q \in \{0, \ldots, M+1\}$, and a couple $(\alpha, \beta) \in I_{q,M+1-q} \setminus \{(\alpha_1, \beta_1)\}$, say (α_2, β_2). Now, applying the operator $\Gamma_{\left(u, \frac{-\alpha_1}{\beta_1} u\right)}$ to

$$\psi_{q,M+1-q,(\alpha_2,\beta_2)}\,(\alpha_2 x + \beta_2 y)\,(x^q, y^{M+1-q})$$

we obtain

$$\begin{aligned}
&\Gamma_{\left(u, \frac{-\alpha_1}{\beta_1} u\right)} \left[\psi_{q,M+1-q,(\alpha_2,\beta_2)}\,(\alpha_2 x + \beta_2 y)\,(x^q, y^{M+1-q})\right] = \\
&\psi_{q,M+1-q,(\alpha_2,\beta_2)} \left(\alpha_2(x+u) + \beta_2\left(y - \frac{\alpha_1}{\beta_1}u\right)\right) \\
&\left((x+u)^q, \left(y - \frac{\alpha_1}{\beta_1}u\right)^{M+1-q}\right) - \\
&\psi_{q,M+1-q,(\alpha_2,\beta_2)}\,(\alpha_2 x + \beta_2 y)\,(x^q, y^{M+1-q}) = \\
&\psi_{q,M+1-q,(\alpha_2,\beta_2)} \left(\alpha_2 x + \beta_2 y + \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\beta_1}u\right) \\
&\left((x+u)^q, \left(y - \frac{\alpha_1}{\beta_1}u\right)^{M+1-q}\right) - \\
&\psi_{q,M+1-q,(\alpha_2,\beta_2)}\,(\alpha_2 x + \beta_2 y)\,(x^q, y^{M+1-q}) = \\
&\sum_{(s',t') \in \{0,\cdots,q\} \times \{0,\cdots,M+1-q\} \setminus \{(0,0)\}} \psi_{q,M+1-q,(\alpha_2,\beta_2)} \left(\alpha_2 x + \beta_2 y + \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\beta_1}u\right) \\
&\left(x^{q-s'}, u^{s'}, y^{M+1-q-t'}, \left(\frac{-\alpha_1}{\beta_1}u\right)^{t'}\right) + \\
&\Delta_{\frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\beta_1}u}\psi_{q,M+1-q,(\alpha_2,\beta_2)}\,(\alpha_2 x + \beta_2 y)\,(x^q, y^{M+1-q}).
\end{aligned}$$

Observe that the first group of summands are polynomials of degree $\leqslant M$, and only the last summand is a polynomial of degree
$q + M + 1 - q = M + 1$, with the coefficients $\hat{\psi}_{q,M+1-q} = \Delta_{\frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\beta_1}u}\psi_{q,M+1-q}$.
Now, let us apply the operator $\Gamma_{(u, \frac{-\alpha_2}{\beta_2}u)}$ to the right-hand side. As we saw before it does not increase the degree of all but the last summand, and in the latter case we obtain (as previously) the vanishing of terms

$$\psi_{q,M+1-q,(\alpha_2,\beta_2)}\,(\alpha_2 x + \beta_2 y)\,(x^q, y^{M+1-q}).$$

Obviously, this can be extended to the case of arbitrary $(\alpha, \beta) \in I_{q,M+1-q}$ for $q \in \{0, \cdots, M+1\}$ in an analogous way.
Therefore, applying the operators $\Gamma_{\left(u, \frac{-\alpha_2}{\beta_2}u\right)}, \cdots, \Gamma_{\left(u, \frac{-\alpha_m}{\beta_m}u\right)}$ to both sides of (E_{M+1}) we get on the left-hand side

$\Delta_u^m \varphi_N(x)(y^N)+$ polynomial in y of degree $< N$,

and on the right-hand side a sum of polynomial functions in (x, y) of degree not exceeding M.

6. By induction hypothesis we infer that $\Delta_u^m \varphi_N$ is locally polynomial of an order at most

$$n = \sum_{m=0}^{M} \text{card}\left(\bigcup_{s=m}^{M} K_s\right) - 1$$

on $\frac{1}{r}\mathcal{K}$, for arbitrary $u \in \frac{1}{r}\mathcal{K}$. Hence, we infer that there exists a $\tilde{p} \in \mathbb{N}$ such that for any $(x, v) \in X \times X$ with

$$x, x+v, \cdots, x+(n+1)v \in \frac{1}{\tilde{p}r}\mathcal{K},$$

and every $u \in \frac{1}{r}\mathcal{K}$ the equality

$$\Delta_v^{n+1}\Delta_u^m \varphi_N(x) = 0 \tag{**}$$

holds. By the absolute convexity of $\mathcal{K}$ we have the following property, if

$$x, x+v, \cdots, x+(n+m+1)v \in \frac{1}{\tilde{p}r}\mathcal{K}$$

then

$$v \in \frac{1}{n+m+1}\left(\frac{1}{\tilde{p}r}\mathcal{K} - X\right) \subset \frac{1}{\tilde{p}r}\mathcal{K} \subset \frac{1}{r}\mathcal{K}.$$

Setting $u = v$ in $(**)$ we obtain

$$\Delta_v^{n+m+1}\varphi_N(x) = \Delta_v^{n+1}\Delta_v^m \varphi_N(x) = 0.$$

It follows that φ_N is locally polynomial of order at most

$$n + m = \sum_{m=0}^{M} \text{card}\left(\bigcup_{s=m}^{M} K_s\right) - 1$$

on the set $\frac{1}{p'}\mathcal{K}$ where $p' = \tilde{p}r$.

□

Now note that in Lemma (6.1.1), if

$$\bigcup_{s=0}^{M} K_s = \left\{(\alpha, \beta) \in \bigcup_{s=0}^{M}\bigcup_{p=0}^{s} I_{p,s-p} : \alpha \in [0,1),\ \beta = 1-\alpha\right\},$$

i.e. $\alpha x + \beta y$ is a convex combination of x and y in ($\mathrm{E_M}$), then we can weaken the assumption on the set $\mathcal{K}$ by not requiring that $0 \in \text{algint}\mathcal{K}$. This is stated in the following corollary.

Corollary 6.1.1. *(cf. Corollary 1.2 in [29]) Let X and Y be two linear spaces over a field $\mathbb{K} \subset \mathbb{R}$ and let $\emptyset \neq \mathcal{K} \subset X$ be a convex set such that $x_0 \in \text{algint}\mathcal{K}$. Fix $N \in \mathbb{N} \cup \{0\}$ and $M \in \mathbb{N} \cup \{-1, 0\}$. Further, if $M \geqslant 0$, assume that for each $m \in \{0, \cdots, M\}$ and each $p \in \{0, \cdots, m\}$, the set $I_{p,m-p} \subset [0,1) \cap \mathbb{K}$ is finite.*

If the functions $\varphi_i : \mathcal{K} \to SA^i(X;Y)$, $i \in \{0, \cdots, N\}$ *and, if* $M \geqslant 0$, $\psi_{p,m-p,\alpha} : \mathcal{K} \to SA^m(X;Y)$, $\alpha \in I_{p,m-p}$, $0 \leqslant p \leqslant m, m \in \{0, \ldots, M\}$ *satisfy*

$$\sum_{i=0}^{N} \varphi_i(x)(y^i) = \begin{cases} 0, & M = -1, \\ \sum_{m=0}^{M} \sum_{p=0}^{m} \sum_{\alpha \in I_{p,m-p}} \psi_{p,m-p,\alpha}\left(\alpha x + (1-\alpha)y\right)(x^p, y^{m-p}), & M \geqslant 0 \end{cases} \tag{6.1.1}$$

for every $x, y \in \mathcal{K}$, *then there exists a convex subset* $\mathcal{K}' \subset \mathcal{K}$ *such that* $x_0 \in \operatorname{algint}\mathcal{K}'$, *and* φ_N *is a locally polynomial function of order at most equal to*

$$\sum_{m=0}^{M} \operatorname{card}\left(\bigcup_{s=m}^{M} K_s\right) - 1,$$

on $\mathcal{K}'$, *where* $K_s = \bigcup_{p=0}^{s} I_{p,s-p}$ *for each* $s \in \{0, \cdots, M\}$, *if* $M \geqslant 0$. *Moreover, if* $M = -1$,

$$\sum_{i=0}^{N} \varphi_i(x)(y^i) = 0$$

then φ_N *is the zero function.*

Using the results of R. Ger from [14] and taking into account representation of polynomial functions (cf. [35]) we get also

Corollary 6.1.2. *(cf. Corollary 1.3 in [29]) Under the assumptions of Corollary 6.1.1 there exist a convex subset* $\mathcal{K}' \subset \mathcal{K} \subset X$ *such that* $x_0 \in \operatorname{algint}\mathcal{K}'$, *and functions* $A_i \in SA^i(X;Y)$, $i \in \{0, \cdots, \ell\}$, *such that for every* $x \in \mathcal{K}'$

$$\varphi_N(x) = A_0 + A_1(x) + A_2^*(x) + \cdots + A_\ell^*(x),$$

where

$$\ell = \sum_{m=0}^{M} \operatorname{card}\left(\bigcup_{s=m}^{M} K_s\right) - 1,$$

and $K_s = \bigcup_{p=0}^{s} I_{p,s-p}$ *for each* $s \in \{0, \cdots, M\}$. *The functions* A_i, $i \in \{0, \cdots, \ell\}$ *are defined uniquely and* $A_\ell^*(x) = A_\ell\underbrace{(x, \cdots, x)}_{\ell \text{ times}}$.

6.2 Applications.

6.2.1 Local equation of Fechner-Gselmann .

Let's see how the machinery works and consider again the Fechner-Gselmann equation

$$F(x+y) - F(x) - F(y) = xf(y) + yf(x) \tag{6.2.1}$$

which was supposed to hold for all $x, y \in \mathbb{R}$.
If we assume that $x, y \in \mathcal{K} \subset X-$ a linear space over the field $\mathbb{K} \subset \mathbb{R}$ and introduce the following substitutions:

$$\tilde{F}(x) = F(2x),$$

(which is defined on $\frac{1}{2}\mathcal{K}$), then it is easy to see that (6.2.1) is equivalent to

$$\tilde{F}\left(\frac{x+y}{2}\right) - \tilde{F}\left(\frac{x}{2}\right) - \tilde{F}\left(\frac{y}{2}\right) = xf(y) + yf(x), \tag{6.2.2}$$

but now quite a natural domain is an interval, or convex subset of $\mathbb{R}$.
Let us rewrite (6.2.2) in the form

$$f(x)y + \hat{F}(x) = -f(y)x - \hat{F}(y) + \tilde{F}\left(\frac{x+y}{2}\right), \tag{6.2.3}$$

where $\hat{F}(x) = \tilde{F}(\frac{x}{2})$. Let us observe that (6.2.3) is a special case of ($\mathrm{E_M}$). Now its enough to take $N = 1 = M$, $\varphi_1 = f$, $\varphi_0 = \hat{F}$, $I_{0,1} = \emptyset$, $I_{0,0} = \{(\frac{1}{2}id, \frac{1}{2}id), (0, id)\}$, $I_{1,0} = \{(0, id)\}$, $\psi_{0,0,(\frac{1}{2},\frac{1}{2})} = \tilde{F}$, $\psi_{0,0,(0,1)} = -\hat{F}$, $\psi_{1,0,(0,1)} = -f$. Then $K_0 = I_{0,0} = \{(\frac{1}{2}id, \frac{1}{2}id), (0, id)\}, K_1 = I_{1,0} \cup I_{0,1} = \{(0, id)\}$, $K_0 \cup K_1 = \{(\frac{1}{2}id, \frac{1}{2}id), (0, id)\}$, hence, by Lemma 6.1.1 we get that f is a locally polynomial function of order at most $\mathrm{card}(K_0 \cup K_1) + \mathrm{card}(K_1) - 1 = 2 + 1 - 1 = 2$, on the set $\frac{1}{2}\mathcal{K}$, and so $\ell = 2$ now using Corollary 6.1.2 we infer that

$$f(x) = A_0 + A_1(x) + A_2^*(x),$$

for all $x \in \frac{1}{2}\mathcal{K}$, where $A_i \in SA^i(X;Y)$, $i \in \{0, \cdots, 2\}$. As in Chapter 3 (cf. [25]) we can also assume that $\hat{F}$ is a locally polynomial functions of order at most 3, hence by Corollary 6.1.2 we have that

$$\hat{F}(x) = \hat{B}_0 + \hat{B}_1(x) + \hat{B}_2^*(x) + \hat{B}_3^*(x),$$

for all $x \in \frac{1}{2}\mathcal{K}$, where $\hat{B}_i \in SA^i(X;Y)$, $i \in \{0, \cdots, 3\}$. Thus, it follows that $\tilde{F}$ is also a locally polynomial function and can be written in the form

$$\tilde{F}(x) = B_0 + B_1(x) + B_2^*(x) + B_3^*(x)$$

for all $x \in \mathcal{K}' := \frac{1}{4}\mathcal{K} \subset \frac{1}{2}\mathcal{K} \subset \mathcal{K}$, where $B_i^*(x) = \hat{B}_i^*(2x)$, $B_i \in SA^i(X;Y)$, $i \in \{0, \cdots, 3\}$.
Now using similar argument in Chapter 3 (cf. [25]), we obtain the local solution of (6.2.2)

a) $f(x) = A_1(x) + a_2x^2$,

b) $\tilde{F}(x) = B_1(x) + 4xA_1(x) + \frac{8}{3}a_2x^3$,

for all $x \in \mathcal{K}'$, where A_1, B_1 are additive functions and a_2 is an arbitrary constant. In [30] one can find techniques of extending the solution from $\mathcal{K}'$ to the whole set $\mathcal{K}$, except may be for the boundary.

6.2.2 J. Ger's equation.

Joanna Ger in [13] considered the following equations, assumed to hold for all $x, y \in I$, where $I \subset \mathbb{R}$ is a non-empty interval.

$$f(x) - f(y) = (x - y)\left[h\left(\frac{x+y}{2}\right) + k(x) + k(y)\right] \tag{6.2.4}$$

and

$$xf(y) - yf(x) = (x - y)\left[h\left(\frac{x+y}{2}\right) + k(x) + k(y)\right], \tag{6.2.5}$$

where $f, h, k : I \to \mathbb{R}$ are unknown. Equations (6.2.4) and (6.2.5) are analogous to equations considered by T. Riedel and P. K. Sahoo in the case where $I = \mathbb{R}$. Solving (6.2.4) and (6.2.5) was crucial for answering the so called M. Merkle's problem, formulated in [24] and reformulated in the language of functional equations by J. Walorski in [38]. Walorski actually was solving the equation

$$g(x) - g(y) = (x - y)\left[\alpha f\left(\frac{x+y}{2}\right) + f(x) + f(y)\right],$$

where $\alpha > 0$. However, he was assumming high regularity of the searched function $g : I \to \mathbb{R}$. J. Ger was able to obtain the general solution of a more general equation (6.2.4) with no regularity requirements.
Now substituting $x = \frac{u+v}{2}, y = \frac{u-v}{2}$ into (6.2.4) we have

$$\hat{h}(u)v = f\left(\frac{u+v}{2}\right) - f\left(\frac{u-v}{2}\right) - \left[k\left(\frac{u+v}{2}\right) + k\left(\frac{u-v}{2}\right)\right]v \tag{6.2.6}$$

where $\hat{h}(u) = h(\frac{u}{2})$. Let us observe that (6.2.6) is a special case of (E_M). Now its enough to take $N = 1 = M$, $\varphi_1 = \hat{h}$, $\varphi_0 = 0$, $I_{1,0} = \emptyset, I_{0,0} = \{(\frac{1}{2}id, \frac{1}{2}id), (\frac{1}{2}id, -\frac{1}{2}id)\} = I_{0,1}$, $\psi_{0,0,(\frac{1}{2},\frac{1}{2})} = f$, $\psi_{0,0,(\frac{1}{2},-\frac{1}{2})} = -f$, $\psi_{0,1,(\frac{1}{2},\frac{1}{2})} = \psi_{0,1,(\frac{1}{2},-\frac{1}{2})} = -k$. Then $K_0 = I_{0,0} = \{(\frac{1}{2}id, \frac{1}{2}id), (\frac{1}{2}id, -\frac{1}{2}id)\}$ $K_1 = I_{1,0} \cup I_{0,1} = \{(\frac{1}{2}id, \frac{1}{2}id), (\frac{1}{2}id, -\frac{1}{2}id)\}$, $K_0 \cup K_1 = \{(\frac{1}{2}id, \frac{1}{2}id), (\frac{1}{2}id, -\frac{1}{2}id)\}$, hence, by Lemma 6.1.1 we get that $\hat{h}$ is a locally polynomial function of order at most $\text{card}(K_0 \cup K_1) + \text{card}(K_1) - 1 = 2 + 2 - 1 = 3$, on the set $\frac{1}{p'_1}I$, for a $p'_1 \in \mathbb{N}$, and so $\ell = 3$ now using Corollary 6.1.2 we infer that

$$\hat{h}(u) = \hat{A}_3^*(u) + \hat{A}_2^*(u) + \hat{A}_1(u) + \hat{A}_0$$

for all $u \in \frac{1}{p'_1}I$, $p'_1 \in \mathbb{N}$, where $\hat{A}_i \in SA^i(X;Y)$, $i \in \{0, \cdots, 3\}$. Hence it follows that h is also a locally polynomial function and can be written in the form

$$h(u) = A_3^*(u) + A_2^*(u) + A_1(u) + A_0 \tag{6.2.7}$$

for all $u \in \frac{1}{2p'_1}I$, $p'_1 \in \mathbb{N}$, where $A_i^*(u) = \hat{A}_i^*(2u)$, $A_i \in SA^i(X;Y)$, $i \in \{0, \cdots, 3\}$.
On the other hand, regrouping equation (6.2.4) we get

$$k(x)y - k(x)x + f(x) = f(y) + \left[h\left(\frac{x+y}{2}\right) + k(y)\right](x - y) \tag{6.2.8}$$

Let us observe that (6.2.8) is a special case of (E_M). Now applying Lemma 6.1.1 and Corollary 6.1.2 again we obtain that

$$k(x) = B_3^*(x) + B_2^*(x) + B_1(x) + B_0 \tag{6.2.9}$$

for all $x \in \frac{1}{p'_2}I$, $p'_2 \in \mathbb{N}$, where $B_i \in SA^i(X;Y)$, $i \in \{0, \cdots, 3\}$.
Let $p' = \max\{p'_1, p'_2\}$, and let us put $y = 0$ in (6.2.4) to calculate f we have

$$\begin{aligned} f(x) = & \\ & x\left[\frac{1}{8}A_3^*(x) + B_3^*(x) + \frac{1}{4}A_2^*(x) + \right. \\ & \left. B_2^*(x) + \frac{1}{2}A_1(x) + B_1(x) + A_0 + 2B_0\right] + f(0) \end{aligned} \tag{6.2.10}$$

for all $x, y \in I' = \frac{1}{p'}I$.
Inserting (6.2.7), (6.2.9) and (6.2.10) into (6.2.4) we obtain the equality linking A_ℓ and $B_\ell, \ell \in \{1, 2, 3\}$, (the constants are reduced). Comparing the summands of the same degree, we get the following system of equations.

$$\frac{1}{8}xA_3^*(y) + xB_3^*(y) = \frac{3}{8}yA_3(x, y, y), \tag{6.2.11}$$

$$\frac{1}{8}yA_3^*(x) + yB_3^*(x) = \frac{3}{8}xA_3(x, x, y), \tag{6.2.12}$$

$$\frac{3}{8}xA_3(x, y, y) = \frac{3}{8}yA_3(x, x, y), \tag{6.2.13}$$

$$\frac{1}{4}xA_2^*(y) + xB_2^*(y) = \frac{1}{2}yA_2(x,y), \tag{6.2.14}$$

$$\frac{1}{4}yA_2^*(x) + yB_2^*(x) = \frac{1}{2}xA_2(x,y), \tag{6.2.15}$$

$$\frac{1}{2}xA_1(y) + xB_1(y) = \frac{1}{2}yA_1(x) + yB_1(x). \tag{6.2.16}$$

From the equation (6.2.16) we infer the existence of a constant $c \in \mathbb{R}$, such that $\frac{1}{2}A_1(x) + B_1(x) = cx, x \in I'$. Denoting $A(x) = \frac{1}{2}A_1(x)$, we easily see that

$$A_1(x) = 2A(x) \tag{6.2.17}$$

$$B_1(x) = -A(x) + cx, \tag{6.2.18}$$

for every $x \in I'$. Now, if we put $x = y$ in (6.2.14) or (6.2.15), we see that

$$4B_2^*(x) = A_2^*(x),$$

for $x \in I'$. Let us insert this equality into (6.2.14) and let us multiply by $\frac{1}{4}$, side by side. We will get after some obvious calculations

$$x^2A_2^*(y) = xyA_2(x,y) = yxA_2(y,x) = y^2A_2^*(x),$$

Now put $y = 1$, we get $A_2^*(x) = bx^2$ for all $x \in I'$, where $b = A_2^*(1) \in \mathbb{R}$, whence it follows that for a real constant b, we have

$$A_2^*(x) = 4bx^2 \tag{6.2.19}$$

$$B_2^*(x) = bx^2, \tag{6.2.20}$$

for all $x \in I'$. Finally, let us consider (6.2.11), (6.2.12) and (6.2.13). First, from (6.2.11) or (6.2.12), we get

$$4B_3^*(x) = A_3^*(x), \tag{6.2.21}$$

for all $x \in I'$. (We put $x = y$, multiply both sides by 8 and reduce the identical summands on both sides). Now, we may substitute B_3^* into (6.2.11); after suitable reductions we may replace the equalities (6.2.11) and (6.2.13) with the equalities

$$xA_3^*(y) = yA_3(x,y,y), \tag{6.2.22}$$

$$xA_3(x,y,y) = yA_3(y,x,x). \tag{6.2.23}$$

Using now the equalities (6.2.22), (6.2.23) and symmetry of A_3, we obtain for all $x, y \in I'$,

$$x^3A_3^*(y) = x^2yA_3(x,y,y) = y^2xA_3(y,x,x) = y^3A_3^*(x), \tag{6.2.24}$$

Now put $y = 1$, we see that $A_3^*(x) = A_3^*(1)x^3$ for all $x \in I'$, $A_3^*(1) \in \mathbb{R}$. Let $A_3^*(1) = 8a$, $a \in \mathbb{R}$. Then in view of (6.2.21) we get

$$A_3^*(x) = 8ax^3 \tag{6.2.25}$$

$$B_3^*(x) = 2ax^3, \tag{6.2.26}$$

for all $x \in I'$. Let us denote $\beta := k(0) = B_0$, $\alpha := -\beta - A_0$. Then

$$A_0 + 2B_0 = -\beta - \alpha + 2\beta = \beta - \alpha.$$

Moreover, let $d := f(0)$. Then for all $x, y \in I'$ we obtain

$$\begin{cases} f(x) = 3ax^4 + 2bx^3 + cx^2 + (\beta - \alpha)x + d, & x \in I', \\ h(x) = 8ax^3 + 4bx^2 + 2A(x) - (\beta + \alpha), & x \in \text{int}I', \\ k(x) = 2ax^3 + bx^2 - A(x) + cx + \beta, & x \in I'. \end{cases}$$

The above solution may be extended to the whole interval I, with exception for the function h, which may be defined arbitrarily on the boundary of I, if $I = \text{cl}I$.
Summarizing, using our "machinery" we may obtain the result by J. Ger from [13] (Theorem 1):

Theorem 6.2.1. *(cf. Theorem 1 in [13] and Theorem 2.1 in [29]) Assume that $I \subset \mathbb{R}$ is an interval with positive length. Functions $f, h, k : I \to \mathbb{R}$ satisfy the equation*

$$f(x) - f(y) = (x - y)\left[h\left(\frac{x+y}{2}\right) + k(x) + k(y)\right]$$

if, and only if

$$\begin{cases} f(x) = 3ax^4 + 2bx^3 + cx^2 + (\beta - \alpha)x + d, & x \in I, \\ h(x) = 8ax^3 + 4bx^2 + 2A(x) - (\beta + \alpha), & x \in \text{int}I, \\ k(x) = 2ax^3 + bx^2 - A(x) + cx + \beta, & x \in I, \end{cases}$$

where A is additive, and $a, b, c, d, \alpha, \beta$ are real constants. If I is closed, then h may be arbitrary on the boundary of I.

6.2.3 Local case of Okeke's equation.

In Chapter 5 (cf. [26]) we obtained the polynomial solutions of following generalized functional equation (6.2.27), under some mild assumptions on the parameters involved

$$\sum_{i=1}^{n} \gamma_i F(a_i x + b_i y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y), \tag{6.2.27}$$

for every $x, y \in \mathbb{R}$, $\gamma_i, \alpha_j, \beta_j \in \mathbb{R}$, and $a_i, b_i, c_j, d_j \in \mathbb{Q}$. Here, we will consider a derivation of (6.2.27) namely,

$$\sum_{i=1}^{n} \gamma_i F(\lambda_i x + (1 - \lambda_i)y) = \sum_{j=1}^{m} \theta_j (\alpha_j x + (1 - \alpha_j)y) f(\beta_j x + (1 - \beta_j)y), \tag{6.2.28}$$

for every $x, y \in \mathcal{K} \subset X$-a linear space over a field $\mathbb{K} \subset \mathbb{R}$, $\mathcal{K}$ being a $\mathbb{K}$-convex subset of X, $\gamma_i, \theta_j \in \mathbb{K}$, and $\lambda_i, \alpha_j, \beta_j, \in [0, 1] \cap \mathbb{K}, i = 1, \cdots, n, j = 1, \cdots, m$.

We begin by showing that (6.2.28) has locally polynomial functions as solutions.

Lemma 6.2.1. *(cf. Lemma 2.1 in [29]) Suppose that X is a linear space over the field $\mathbb{K} \subset \mathbb{R}$, let $\emptyset \neq \mathcal{K} \subset X$ be a convex set, let $\gamma_i, \theta_j \in \mathbb{K}$, and $, \lambda_i, \alpha_j, \beta_j, \in [0, 1] \cap \mathbb{K}, i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}$. Suppose further that the functions $F, f : \mathcal{K} \to \mathbb{K}$ satisfy equation (6.2.28). If there exists $j_0 \in \{1, \cdots, m\}$ with $\theta_{j_0} \neq 0$ such that*

$$\det \begin{pmatrix} \alpha_{j_0} & 1 - \alpha_{j_0} \\ \beta_{j_0} & 1 - \beta_{j_0} \end{pmatrix} \neq 0, \tag{6.2.29}$$

$$\det \begin{pmatrix} \lambda_i & 1 - \lambda_i \\ \beta_{j_0} & 1 - \beta_{j_0} \end{pmatrix} \geqslant 0, \tag{6.2.30}$$

and

$$\det \begin{pmatrix} \beta_j & 1-\beta_j \\ \beta_{j_0} & 1-\beta_{j_0} \end{pmatrix} \geqslant 0, \tag{6.2.31}$$

for every $i = 1, \cdots, n, j = 1, \cdots, m$, then f is a locally polynomial function of order at most equal to $k-1$, on the set $\mathcal{K}' \subset \mathcal{K}$, where k is the total number of distinct numbers λ_i, β_j.

Proof. Suppose that the pair (F, f) satisfies (6.2.28), fix a $j_0 \in \{1, \cdots, m\}$ with $\theta_{j_0} \neq 0$ such that (6.2.29),(6.2.30) and, (6.2.31) holds. Rewrite (6.2.28) in the following form

$$\begin{aligned} \sum_{i=1}^{n} \gamma_i F(\lambda_i x + (1-\lambda_i)y) &= \sum_{j \neq j_0} \theta_j(\alpha_j x + (1-\alpha_j)y) f(\beta_j x + (1-\beta_j)y), \\ &\quad + \theta_{j_0}(\alpha_{j_0} x + (1-\alpha_{j_0})y) f(\beta_{j_0} x + (1-\beta_{j_0})y) \end{aligned} \tag{6.2.32}$$

if $\beta_{j_0} = 1$ then we may use Corollary (6.1.1) directly. Now assume that $\beta_{j_0} \in [0,1) \cap \mathbb{K}$, let $z = \beta_{j_0} x + (1-\beta_{j_0})y$ and $w = x$ and substitute in (6.2.32) we obtain

$$\begin{aligned} &\sum_{i=1}^{n} \gamma_i F\left(\lambda_i w + (1-\lambda_i)\left(\tfrac{z-\beta_{j_0} w}{1-\beta_{j_0}}\right)\right) \\ &\quad = \sum_{j \neq j_0} \theta_j \left(\alpha_j w + (1-\alpha_j)\left(\tfrac{z-\beta_{j_0} w}{1-\beta_{j_0}}\right)\right) f\left(\beta_j w + (1-\beta_j)\left(\tfrac{z-\beta_{j_0} w}{1-\beta_{j_0}}\right)\right) \\ &\quad + \theta_{j_0}\left(\alpha_{j_0} w + (1-\alpha_{j_0})\left(\frac{z-\beta_{j_0} w}{1-\beta_{j_0}}\right)\right) f(z) \end{aligned}$$

and so,

$$\begin{aligned} &f(z)\left(\tfrac{\alpha_{j_0}-\beta_{j_0}}{1-\beta_{j_0}}\right)\theta_{j_0} w + f(z)\left(\tfrac{1-\alpha_{j_0}}{1-\beta_{j_0}}\right)\theta_{j_0} z \\ &\quad = \sum_{i=1}^{n} \gamma_i F\left(\left(1 - \tfrac{\lambda_i-\beta_{j_0}}{1-\beta_{j_0}}\right) z + \left(\tfrac{\lambda_i-\beta_{j_0}}{1-\beta_{j_0}}\right) w\right) \\ &\quad - \sum_{j \neq j_0} \theta_j \left(\tfrac{1-\alpha_j}{1-\beta_{j_0}}\right) z f\left(\left(1 - \tfrac{\beta_j-\beta_{j_0}}{1-\beta_{j_0}}\right) z + \left(\tfrac{\beta_j-\beta_{j_0}}{1-\beta_{j_0}}\right) w\right) \\ &\quad - \sum_{j \neq j_0} \theta_j \left(\tfrac{\alpha_j-\beta_{j_0}}{1-\beta_{j_0}}\right) w f\left(\left(1 - \tfrac{\beta_j-\beta_{j_0}}{1-\beta_{j_0}}\right) z + \left(\tfrac{\beta_j-\beta_{j_0}}{1-\beta_{j_0}}\right) w\right) \end{aligned}$$

Using (6.2.29) we see that $\alpha_{j_0} \neq \beta_{j_0}$, so the coefficient $(\frac{\alpha_{j_0}-\beta_{j_0}}{1-\beta_{j_0}})\theta_{j_0} w$ of $f(z)$ is different from zero. Now by (6.2.30), (6.2.31) we get that $\lambda_i, \beta_j \geqslant \beta_{j_0}$, and since $0 \leqslant, \lambda_i, \beta_j < 1$, therefore we have that $\frac{\lambda_i-\beta_{j_0}}{1-\beta_{j_0}}, \frac{\beta_j-\beta_{j_0}}{1-\beta_{j_0}} \in [0,1) \cap \mathbb{K}$ for every $i = 1, \cdots, n, j = 1, \cdots, m$. Hence, by Corollary (6.1.1), f is a locally polynomial function on the set $\mathcal{K}' \subset \mathcal{K}$, of order at most

$$\operatorname{card}(K_0 \cup K_1) + \operatorname{card}(K_1) - 1 = k - 1$$

where $K_0 = I_{0,0} = \{\frac{\lambda_i-\beta_{j_0}}{1-\beta_{j_0}}, i = 1, \cdots, n\}$, $K_1 = I_{1,0} \cup I_{0,1} = \{\frac{\beta_j-\beta_{j_0}}{1-\beta_{j_0}}, j \neq j_0\}$, and k is the total number of distinct numbers λ_i, β_j in the set $K_0 \cup K_1$, and K_1. □

6.2.4 Non-symmetric version of Okeke's equation.

From the results of Chapter 3, 4 and 5 (see [25], [28] and [26]), we may assume that also F in equation (6.2.28) is a locally polynomial function.

Now let us assume that (6.2.28) is satisfied for real x, y from an interval $\mathcal{K}$ of $\mathbb{R}$ which additionally we assume to be symmetric around 0. Let us consider the case $x \neq y \in \mathcal{K}$. Without loss of generality let $x < y$. Then there exist a $z \in \mathcal{K}$ such that $y = x + z$. Now substituting into equation (6.2.28) we have

$$\sum_{i=1}^{n} \gamma_i F(x + (1-\lambda_i)z) = \sum_{j=1}^{m} \theta_j (x + (1-\alpha_j)z) f(x + (1-\beta_j)z), \tag{6.2.33}$$

It is equation (6.2.33) we are going to solve.

A characteristic feature of (6.2.33) is the dependence of the existence of solutions on the behaviour of the sequences l_k and r_k given by

$$l_k = \sum_{i=1}^{n} \gamma_i (2-\lambda_i)^{k+1} \tag{6.2.34}$$

and,

$$r_k = \sum_{j=1}^{m} \theta_j (2-\alpha_j)(2-\beta_j)^k \tag{6.2.35}$$

respectively, for all $k \in \mathbb{N} \cup \{0\}$.

Theorem 6.2.2. *(cf. Theorem 2.2 in [29]) Let $\mathcal{K}$ be a non-empty convex set of a linear space X over a field $\mathbb{K} \subset \mathbb{R}$. Let $k \in \mathbb{N} \cup \{0\}$ and $\gamma_i, \theta_j \in \mathbb{K}, , \lambda_i, \alpha_j, \beta_j, \in [0,1] \cap \mathbb{K} \subset \mathbb{Q}, i = 1, \cdots, n, j = 1, \cdots, m$ be such that $l_k, r_k \neq 0$ is given by (6.2.34) and (6.2.35) respectively. Suppose further that equation (6.2.33)(hence of (6.2.28)) is satisfied by the pair $(F, f) : \mathcal{K} \to \mathbb{K}$ of monomial functions of order $k+1$ and k respectively.*

(a) if $k = 0$ then $f = 0 = F$ or $f = A_0 \neq 0$ and $F(x) = \frac{r_0}{l_0} A_0 x$; in the latter case necessarily

$$\frac{r_0}{l_0} \sum_{i=1}^{n} \gamma_i = \sum_{j=1}^{m} \theta_j, \tag{6.2.36}$$

and,

$$\frac{r_0}{l_0} \sum_{i=1}^{n} \gamma_i (1-\lambda_i) = \sum_{j=1}^{m} \theta_j (1-\alpha_j). \tag{6.2.37}$$

(b) if $k \geqslant 1$ then $f = 0 = F$, or $f(x) = A_k^*(x) = A_k \underbrace{(x, \cdots, x)}_{k \text{ times}} \neq 0$ is an arbitrary $k-$additive symmetric function and $F(x) = \frac{r_k}{l_k} x f(x), x \in \mathcal{K}$.
Moreover for $p \in \{1, \cdots, k\}$, if $f \neq 0$ and the below equations

$$\frac{r_k}{l_k} \sum_{i=1}^{n} \gamma_i = \sum_{j=1}^{m} \theta_j, \tag{6.2.38}$$

$$\frac{r_k}{l_k} \sum_{i=1}^{n} \gamma_i (1-\lambda_i)^{k+1} = \sum_{j=1}^{m} \theta_j (1-\alpha_j)(1-\beta_j)^k, \tag{6.2.39}$$

and,

$$\frac{r_k}{l_k} \sum_{i=1}^{n} \binom{k+1}{p} \gamma_i (1-\lambda_i)^p = \sum_{j=1}^{m} \binom{k}{p-1} \theta_j (1-\alpha_j)(1-\beta_j)^{p-1} + \sum_{j=1}^{m} \binom{k}{p} \theta_j (1-\beta_j)^p, \tag{6.2.40}$$

are satisfied then either

(i) $\sum_{i=1}^{n} \gamma_i (1-\lambda_i)^p = \sum_{j=1}^{m} \theta_j (1-\beta_j)^p$ for each $p \in \{1, \cdots, k\}$, and f is an arbitrary $k-$monomial function or

(ii) $\sum_{i=1}^{n}\gamma_i(1-\lambda_i)^p \neq \sum_{j=1}^{m}\theta_j(1-\beta_j)^p$ *for each* $p\in\{1,\cdots,k\}$*, and* f *is a continuous* k*-monomial function and so* F *is of order* $k+1$*.*

Proof. Let $k=0$ such that $f=const=A_0\neq 0$ and F is additive. Putting $x=z$ in (6.2.33) we have

$$\sum_{i=1}^{n}\gamma_i(2-\lambda_i)F(x)=\sum_{j=1}^{m}\theta_j(2-\alpha_j)xA_0$$

i.e.,

$$F(x)=\frac{r_0}{l_0}xA_0=D_0x \tag{6.2.41}$$

for any $x\in\mathcal{K}$, because $D_0=\frac{r_0}{l_0}A_0\neq 0$ therefore, F is a continuous function. Now substituting (6.2.41) into (6.2.33) we get

$$D_0\sum_{i=1}^{n}\gamma_i(x+(1-\lambda_i)z)=\sum_{j=1}^{m}\theta_j(x+(1-\alpha_j)z)A_0,$$

$$\left(D_0\sum_{i=1}^{n}\gamma_i\right)x+\left(D_0\sum_{i=1}^{n}\gamma_i(1-\lambda_i)\right)z=\left(A_0\sum_{j=1}^{m}\theta_j\right)x+\left(A_0\sum_{j=1}^{m}\theta_j(1-\alpha_j)\right)z$$

for all $x,z\in\mathcal{K}$. Thus nothing that $D_0=\frac{r_0}{l_0}A_0\neq 0$, and comparing the terms with the same degree we obtain (6.2.36) and (6.2.37).
Let $k\geqslant 1$ such that $f(x)=A_k^*(x)$ and $F(x)=\frac{r_k}{l_k}xf(x), x\in\mathcal{K}$, satisfies (6.2.33). Set $D_k=\frac{r_k}{l_k}$, we can write (6.2.33) as

$$D_k\sum_{i=1}^{n}\gamma_i(x+(1-\lambda_i)z)A_k^*(x+(1-\lambda_i)z)=\sum_{j=1}^{m}\theta_j(x+(1-\alpha_j)z)A_k^*(x+(1-\beta_j)z),$$

or,

$$\begin{aligned}&D_k\sum_{i=1}^{n}\gamma_i(x+(1-\lambda_i)z)\left(\sum_{p=0}^{k}\binom{k}{p}(1-\lambda_i)^pA_k(x^{k-p},z^p)\right)\\&=\sum_{j=1}^{m}\theta_j(x+(1-\alpha_j)z)\left(\sum_{p=0}^{k}\binom{k}{p}(1-\beta_j)^pA_k(x^{k-p},z^p)\right),\end{aligned}$$

whence,

$$\begin{aligned}&D_k\sum_{i=1}^{n}\gamma_ixA_k^*(x)+D_k\sum_{i=1}^{n}\gamma_i\left(\sum_{p=1}^{k}\binom{k}{p}(1-\lambda_i)^pxA_k(x^{k-p},z^p)\right)\\&+D_k\sum_{i=1}^{n}\gamma_i(1-\lambda_i)^{k+1}zA_k^*(z)+D_k\sum_{i=1}^{n}\gamma_i\left(\sum_{p=0}^{k-1}\binom{k}{p}(1-\lambda_i)^{p+1}zA_k(x^{k-p},z^p)\right)\\&=\sum_{j=1}^{m}\theta_jxA_k^*(x)+\sum_{j=1}^{m}\theta_j\left(\sum_{p=1}^{k}\binom{k}{p}(1-\beta_j)^pxA_k(x^{k-p},z^p)\right)\\&+\sum_{j=1}^{m}\theta_j(1-\alpha_j)(1-\beta_j)^kzA_k^*(z)+\sum_{j=1}^{m}\theta_j\left(\sum_{p=0}^{k-1}\binom{k}{p}(1-\alpha_j)(1-\beta_j)^pzA_k(x^{k-p},z^p)\right),\end{aligned}$$

or,

$$\begin{aligned}&D_k\sum_{i=1}^{n}\gamma_ixA_k^*(x)+D_k\sum_{i=1}^{n}\gamma_i\left(\sum_{p=1}^{k}\binom{k}{p}(1-\lambda_i)^pxA_k(x^{k-p},z^p)\right)\\&+D_k\sum_{i=1}^{n}\gamma_i(1-\lambda_i)^{k+1}zA_k^*(z)+D_k\sum_{i=1}^{n}\gamma_i\left(\sum_{p=1}^{k}\binom{k}{p-1}(1-\lambda_i)^pzA_k(x^{k-p+1},z^{p-1})\right)\\&=\sum_{j=1}^{m}\theta_j\left(\sum_{p=1}^{k}\binom{k}{p}(1-\beta_j)^pxA_k(x^{k-p},z^p)\right)+\sum_{j=1}^{m}\theta_j(1-\alpha_j)(1-\beta_j)^kzA_k^*(z)\\&+\sum_{j=1}^{m}\theta_jxA_k^*(x)+\sum_{j=1}^{m}\theta_j\left(\sum_{p=1}^{k}\binom{k}{p-1}(1-\alpha_j)(1-\beta_j)^{p-1}zA_k(x^{k-p+1},z^{p-1})\right),\end{aligned}$$

for every $x, z \in \mathcal{K}$. Now comparing terms of equal degrees leads to the following equations

$$\left(D_k \sum_{i=1}^{n} \gamma_i - \sum_{j=1}^{m} \theta_j\right) x A_k^*(x) = 0, \tag{6.2.42}$$

$$\left(D_k \sum_{i=1}^{n} \gamma_i (1-\lambda_i)^{k+1} - \sum_{j=1}^{m} \theta_j (1-\alpha_j)(1-\beta_j)^k\right) z A_k^*(z) = 0, \tag{6.2.43}$$

and,

$$\left(D_k \sum_{i=1}^{n} \binom{k}{p} \gamma_i (1-\lambda_i)^p - \sum_{j=1}^{m} \binom{k}{p} \theta_j (1-\beta_j)^p\right) x A_k(x^{k-p}, z^p) \tag{6.2.44}$$

$$= \left(\sum_{j=1}^{m} \binom{k}{p-1} \theta_j (1-\alpha_j)(1-\beta_j)^{p-1} - D_k \sum_{i=1}^{n} \binom{k}{p-1} \gamma_i (1-\lambda_i)^p\right) z A_k(x^{k-p+1}, z^{p-1}),$$

for $p \in \{1, \cdots, k\}$, and for every $x, z \in \mathcal{K}$. Observe that if (6.2.42),(6.2.43) and (6.2.44) holds then either $A_k = 0$ or the equations (6.2.38),(6.2.39) and (6.2.40) are satisfied. Assume that $A_k = 0$ then $f = F = 0$ is the only solution to (6.2.33). Now let us consider the non trivial solutions of (6.2.33) that is when $A_k \neq 0$ and equations (6.2.38),(6.2.39) and (6.2.40) holds. Substituting (6.2.40) into (6.2.44) we obtain,

$$\left(\sum_{i=1}^{n} \gamma_i (1-\lambda_i)^p - \sum_{j=1}^{m} \theta_j (1-\beta_j)^p\right) \left(x A_k(x^{k-p}, z^p) - z A_k(x^{k-p+1}, z^{p-1})\right) = 0 \tag{6.2.45}$$

for $p \in \{1, \cdots, k\}$, and for every $x, z \in \mathcal{K}$. From (6.2.45) we see that either

$$\sum_{i=1}^{n} \gamma_i (1-\lambda_i)^p = \sum_{j=1}^{m} \theta_j (1-\beta_j)^p$$

for $p \in \{1, \cdots, k\}$, which leads to a situation where $f = A_k \neq 0$ can be an arbitrary k−monomial function and we get that the pair (F, f) is a solution to (6.2.33) or

$$z A_k(x^{k-p+1}, z^{p-1}) = x A_k(x^{k-p}, z^p) \tag{6.2.46}$$

for $p \in \{1, \cdots, k\}$, and for every $x, z \in \mathcal{K}$. Now by (6.2.46) for $p \in \{1, \cdots, k\}$, we obtain

$$z^k A_k^*(x) = z^{k-1} \left[z A_k^*(x)\right] = z^{k-1} \left[x A_k(x^{k-1}, z)\right] = \cdots = x^k A_k^*(z),$$

for every $x, z \in \mathcal{K}$. Put $z = 1$ we get

$$A_k^*(x) = A_k^*(1) x^k,$$

for every $x \in \mathcal{K}$. This means that $f = A_k \neq 0$ is a continuous k−monomial function and so F is of order $k + 1$. □

We note here that in equation (6.2.28) and (6.2.34), if $f = 0$ and $k = 0$ with $\sum_{i=1}^{n} \gamma_i = 0$ then F have some interesting properties. In particular,

Remark 6.2.1. *(cf. Remark 2.1 in [29]) Suppose that in equation (6.2.28) and (6.2.34), $f = 0$, $k = 0$ with $\sum_{i=1}^{n} \gamma_i = 0$,*

a) If $l_0 \neq 0$ then $F(x) = b$

b) If $l_0 = 0$ then $F(x) = A(x) + b$

where A is additive and b a real constant.

Chapter 7

On symbolic computation of C.P. Okeke functional equations using python programming language

In functional equations theory, fewer methods exist for solving a broader class of functional equations. In situations where such a method exists, it requires tedious computations. Therefore, this present chapter is inspired by one of the questions posed in Chapter 5 (see Remark 5.2.7b) and [26](see Remark 2.10b). In particular, we aim to develop a robust computer code based on the theoretical results obtained in Chapter 5(cf. [26]), which determines the polynomial solutions of the following functional equation,

$$\sum_{i=1}^{n} \gamma_i F(a_i x + b_i y) = \sum_{j=1}^{m} (\alpha_j x + \beta_j y) f(c_j x + d_j y), \tag{7.0.1}$$

for every $x, y \in \mathbb{R}$, $\gamma_i, \alpha_j, \beta_j \in \mathbb{R}$, and $a_i, b_i, c_j, d_j \in \mathbb{Q}$, and its special forms. The primary motivation for writing such a computer code is that solving even simple equations belonging to class (7.0.1) needs long and tiresome calculations. Therefore, one of the advantages of such a computer code is that it allows us to solve complicated problems quickly, easily, and efficiently. Additionally, the computer code will significantly improve the level of accuracy in calculations. Along with that, there is also the factor of speed. Furthermore, the computer code will be fully equipped with the functionality to solve various real-life problems, for example, the functional equations stemming from quadrature rules such as the Midpoint rule, Simpson rule, and Trapezoidal rule used in numerical analysis for integral approximation; the functional equations connected to Lagrange mean value theorem which has many applications in mathematical analysis, computational mathematics, and other fields, and the functional equations arising from descriptive geometry, which is still today a rigorous way to deal with graphical constructions. We point out that the computer code will operate with symbolic calculations provided by Python programming language, which means that it does not contain any numerical or approximate methods, and it yields the exact solutions of the equations considered. We acknowledge that using a computer programming language to solve functional equations has been studied by fewer mathematicians. We mention here some of them, S. Baják and Z. Páles [5], and [6], G.G. Borus and A. Gilányi [7], A. Házy [16], and [17], and C.P. Okeke and M. Sablik [28]. In their works, they used Maple as the programming tool to obtain their results which is less flexible in usage and constitutes only a small portion of the academic research community; however, in this chapter, we achieved our results using Python programming language, designed to be an easily readable, highly versatile, general-purpose, open-source, avails robustness and facilitates the deployment of theorems into computational and symbolic frameworks. The special forms of (7.0.1) have been studied by several mathematicians. Let us quote here a few of them, J. Aczél [1], J. Aczél and M. Kuczma [2], C. Alsina, M. Sablik, and J. Sikorska [4], W. Fechner and E.

Gselmann [11], B. Koclęga-Kulpa, T. Szostok and S. Wąsowicz [18], [19] and [20], T. Nadhomi, C. P. Okeke, M. Sablik and T. Szostok [25], and C. P. Okeke and M. Sablik [28].

7.1 Algorithm and Computer code

Given that the proofs of the theoretical results obtained in Chapter 5 (cf. [26]) are constructive, therefore, we can formulate the following algorithm to solve any equation of type (7.0.1).

1) We rewrite the equation to get a form similar to (3.1.1) and apply Lemma 3.1.1 to obtain the potential polynomial degree of one of the unknown functions, either F or f.

2) From Lemma 3.1.1, we get that

   a) If the potential degree of F was obtained, then

$$k = \begin{cases} m-1 & if\, m \geqslant 1, \\ 0 & if\, m = -1, 0 \end{cases} \tag{7.1.1}$$

   whereas if the potential degree of f was obtained, then

$$k = \begin{cases} m & if\, m \geqslant 0, \\ 0 & if\, m = -1. \end{cases} \tag{7.1.2}$$

3) Using k obtained above, we apply Theorem 5.1.1 by checking its conditions.

4) Next, we check if Proposition 5.1.1, Proposition 5.1.2, Remark 5.1.2 and Remark 5.1.3 are satisfied.

5) Finally, we combine the results obtained in steps 3 and 4 to get the exact polynomial solutions of the functions that satisfy equations of class (7.0.1).

7.1.1 Description of the computer code

The Python code described below runs only on the Python Sagemath environment. It is important to note that the line **from sage.all import *** will work only on a Python with the Sagemath installed. Our code steps are as follows:

(a) Import the following python libraries: sage.all, sys, sympy (Function and Symbol), numpy, scipy.special (comb), and time

```
import sys
from sage.all import *
from sympy import Function, Symbol
import numpy as np
from scipy.special import comb
import time
x=Symbol('x')
y=Symbol('y')
f= Function('f')
F=Function('F')
```

(b) We defined a python function called **PSFE** $(\cdot)$(Polynomial Solutions of Functional Equations), where the entire code is embedded and takes in a functional equation of the form:

$$\sum_{i=1}^{n}\gamma_i F(a_i x + b_i y) - \sum_{j=1}^{m}(\alpha_j x + \beta_j y) f(c_j x + d_j y) = 0 \qquad (7.1.3)$$

where "= 0" is not part of the input. That is, the code is executed with the command:

$$PSFE\left(\sum_{i=1}^{n}\gamma_i F(a_i x + b_i y) - \sum_{j=1}^{m}(\alpha_j x + \beta_j y) f(c_j x + d_j y)\right)$$

The input has functions in terms of F or f, where f is multiplied by a variable x and/or y, and we note that any other representation will pop an error message.

(c) The sub-python function **Seperator12** $(\cdot)$ defined in the python function **PSFE** $(\cdot)$ separates the input into an array or list separated by ",".

(d) We defined another python function **Left_right3** $(\cdot)$ in **PSFE** $(\cdot)$ that transforms the list in (c) above into the form of equation (3.1.1). Recognize N and M and apply Lemma 3.1.1 to obtain m, then the value of k is obtained by equations (7.1.1) or (7.1.2). To see the calculation process taken by the code in applying Lemma 3.1.1, we recommend you to see Theorem 3.2.1, Example 3.2.2 and Example 3.2.3 in Chapter 3 and Theorem 5.2.1 - 5.2.4 and Example 5.2.1 - 5.2.4 in Chapter 5.

(e) Note that (c) and (d) above are encapsulated in the **Sablik_Lemma** $(\cdot)$ function contained in **PSFE** $(\cdot)$. Next, we rearrange the functional equation again in the form of equation (7.0.1), obtain the values of n, m and the parameters $\gamma_i, \alpha_j, \beta_j \in \mathbb{R}, a_i, b_i, c_j, d_j \in \mathbb{Q}, i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$. For different values of k, use Theorem 5.1.1 to obtain the desired monomials alongside verifying if any of Proposition 5.1.1, Proposition 5.1.2, Remark 5.1.2 and Remark 5.1.3 apply. The monomials are summed in their generic form to give a polynomial. These processes are contained in the python function:
Theorem_1_3_Proposition_1_4_1_5_Remark_1_2_1_3$(\cdot,\cdot)$.

(f) The processes mentioned in (c), (d) and (e) combines to form **PSFE** $(\cdot)$.

```
def PSFE(Equation):
    start_time = time.time()
    k = Sablik_Lemma(Equation)
    if k == "":
        print('f(x) = 0')
        print('F(x) = 0')
    else:
        Theorem_1_3_Proposition_1_4_1_5_Remark_1_2_1_3(Equation,k)
    print("-%s seconds-" % round(time.time()-start_time,2))
```

You can download the python source code from the below GitHub URL:

https://github.com/CPOkeke/CPOkeke-Polynomial-Solutions-of-Functional-Equations or please send your request to the below e-mail address:

chisom.okeke@us.edu.pl

It is important to note that the developed python codes were based on Python version 3.8, Sagemath 9.2, and all prerequisite requirements therein. Python and Sagemath are open-source programming software and a little adjustment may be required in the future to get the codes running in future versions of the aforementioned software.

7.1.2 Results of the computer code

Functional equations are mostly named after the mathematicians who discovered them. Sometimes, the functional equations are given names based on the property involved in the given functional equation. Our developed python codes only accept inputs written in terms of variables x and/or y and functions F and/or f that belong to the functional equation of class (7.0.1). The following are examples of well-known functional equations of class (7.0.1) solved with the computer code. Suppose that $(F, f) : \mathbb{R} \to \mathbb{R}$,

Example 7.1.1. *Fechner-Gselmann functional equatuon(cf. Theorem 3.1 in [11], Proposition 3.2 in [25], Proposition 3.2.1 in Chapter 3, and Example 2 in [28]).*

$$F(x+y) - F(x) - F(y) = xf(y) + yf(x)$$

for all $x, y \in \mathbb{R}$.
INPUT:
$PSFE(F(x+y) - F(x) - F(y) - x * f(y) - y * f(x))$

OUTPUT:
By Sablik Lemma f has degree at most 2
$f(x) = B_1(x) + a_3x^2$
$F(x) = A_1(x) + xB_1(x) + \frac{1}{3}a_3x^3$
where a_3 *is a real number.*
$A_1(x), B_1(x)$ *are arbitrary additive functions.*
- 0.05 seconds -

Example 7.1.2. *B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz in [19], considered the following functional equations, which stems from a well-known quadrature rule used in numerical analysis.*

a) see Theorem 1 in [19], Theorem 2.4 in [26] and Theorem 5.2.1 in Chapter 5

$$8\left[F(y) - F(x)\right] = (y-x)\left[f(x) + 3f\left(\tfrac{x+2y}{3}\right) + 3f\left(\tfrac{2x+y}{3}\right) + f(y)\right]$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(8*(F(y) - F(x)) - (y-x)*(f(x) + 3*f((x+2*y)*Rational(1/3)) + 3*f((2*x+y)*Rational(1/3)) + f(y)))$

OUTPUT:
By Sablik Lemma f has degree at most 5
$f(x) = a_1 + a_2x + a_3x^2 + a_4x^3$
$F(x) = a_0 + a_1x + \frac{1}{2}a_2x^2 + \frac{1}{3}a_3x^3 + \frac{1}{4}a_4x^4$
where a_0, a_1, a_2, a_3, a_4 *are real numbers.*
- 0.13 seconds -

b) *see Theorem 2 in [19], Theorem 2.5 in [26] and Theorem 5.2.2 in Chapter 5*

$$F(y) - F(x) = (y - x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right]$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(y) - F(x) - (y - x) * (Rational(1/6) * f(x) + Rational(2/3) * f((x + y) * Rational(1/2)) + Rational(1/6) * f(y)))$

OUTPUT:
By Sablik Lemma f has degree at most 3
$f(x) = a_1 + a_2x + a_3x^2 + a_4x^3$
$F(x) = a_0 + a_1x + \frac{1}{2}a_2x^2 + \frac{1}{3}a_3x^3 + \frac{1}{4}a_4x^4$
where a_0, a_1, a_2, a_3, a_4 *are real numbers.*
- 0.09 seconds -

Example 7.1.3. *In [20] B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz, considered the polynomial functions connected with Hermite-Hadamard inequality in the class of continuous functions(cf. Theorem 4 in [20], Theorem 2.7 in [26] and Theorem 5.2.4 in Chapter 5)*

$$F(y) - F(x) = (y - x)\left[\tfrac{1}{4}f(x) + \tfrac{3}{4}f\left(\tfrac{1}{3}x + \tfrac{2}{3}y\right)\right]$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(y) - F(x) - (y - x) * (Rational(1/4) * f(x) + Rational(3/4) * f(Rational(1/3) * x + Rational(2/3) * y)))$

OUTPUT:
By Sablik Lemma f has degree at most 2
$f(x) = a_1 + a_2x + a_3x^2$
$F(x) = a_0 + a_1x + \frac{1}{2}a_2x^2 + \frac{1}{3}a_3x^3$
where a_0, a_1, a_2, a_3 *are real numbers.*
- 0.05 seconds -

Example 7.1.4. *J. Aczél [1], J. Aczél and M. Kuczma [2] considered variations of the Lagrange mean value theorem, which has many applications in mathematical analysis, computational mathematics, and other fields.*

a) *see J. Aczél result in [1], Example 2.2 in [26] and Example 5.2.2 in Chapter 5*

$$\tfrac{F(y)-F(x)}{y-x} = f(x + y)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(y) - F(x) - ((y - x) * f(x + y)))$

OUTPUT:
By Sablik Lemma F has degree at most 2
$f(x) = a_1 + a_2 x$
$F(x) = a_0 + a_1 x + a_2 x^2$
where a_0, a_1, a_2 *are real numbers.*
- 0.02 seconds -

b) see Theorem 5 in [1], Example 2.3 in [26] and Example 5.2.3 in Chapter 5

$$\frac{F(x)-F(y)}{x-y} = f\left(\frac{x+y}{2}\right)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x) - F(y) - ((x - y) * f((x + y) * Rational(1/2))))$

OUTPUT:
By Sablik Lemma F has degree at most 2
$f(x) = a_1 + a_2 x$
$F(x) = a_0 + a_1 x + \frac{1}{2} a_2 x^2$
where a_0, a_1, a_2 *are real numbers.*
- 0.03 seconds -

Example 7.1.5. *C. Alsina, M. Sablik, and J. Sikorska in [4] considered a functional equation arising from descriptive geometry, which is still today a rigorous way to deal with graphical constructions (cf. Example 2.4 in [26] and Example 5.2.4 in Chapter 5).*

$$2F(y) - 2F(x) = (y - x)\left[f\left(\frac{x+y}{2}\right) + \frac{f(x)+f(y)}{2}\right]$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(2*(F(y)-F(x))-(y-x)*(f((x+y)*Rational(1/2))+((f(x)+f(y))*Rational(1/2))))$

OUTPUT:
By Sablik Lemma f has degree at most 3
$f(x) = a_1 + a_2 x$
$F(x) = a_0 + a_1 x + \frac{1}{2} a_2 x^2$
where a_0, a_1, a_2 *are real numbers.*
- 0.09 seconds -

Example 7.1.6. *(cf. Example 2 in [25] and Example 3.2.2 in Chapter 3)*

$$F(x) - 4F\left(\frac{x+y}{2}\right) + F(y) = xf(y) + yf(x)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x) - 4 * F((x + y) * Rational(1/2)) + F(y) - x * f(y) - y * f(x))$

OUTPUT:
By Sablik Lemma f has degree at most 2
$f(x) = B_1(x) + a_1$
$F(x) = -a_1x - xB_1(x)$
where a_1 is a real number.
$B_1(x)$ is an arbitrary additive function.
- 0.05 seconds -

Example 7.1.7. *(cf. Example 3 in [25] and Example 3.2.3 in Chapter 3)*

$$F(x) - 8F\left(\frac{x+y}{2}\right) + F(y) = xf(y) + yf(x)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x) - 8 * F((x + y) * Rational(1/2)) + F(y) - x * f(y) - y * f(x))$

OUTPUT:
By Sablik Lemma f has degree at most 2
$f(x) = a_1 + a_3x^2$
$F(x) = -\frac{1}{3}a_1x - \frac{1}{3}a_3x^3$
where a_1, a_3 are real numbers.
- 0.06 seconds -

Example 7.1.8. *Cauchy additive functional equation*

$$F(x + y) = F(x) + F(y)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x + y) - F(x) - F(y))$

OUTPUT:
By Sablik Lemma F has degree at most 1
$f(x) = 0$
$F(x) = A_1(x)$
$A_1(x)$ is an arbitrary additive function.
- 0.01 seconds -

Example 7.1.9. *Jensen functional equation*

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}\left[F(x) + F(y)\right]$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F((x + y) * Rational(1/2)) - Rational(1/2) * (F(x) + F(y)))$

OUTPUT:
By Sablik Lemma F has degree at most 1
$f(x) = 0$
$F(x) = a_0 + A_1(x)$
where a_0 *is a real number.*
$A_1(x)$ *is an arbitrary additive function.*
- 0.01 seconds -

Example 7.1.10. *Drygas functional equation*

$$F(x+y) + F(x-y) = 2F(x) + F(y) + F(-y)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x+y) + F(x-y) - 2 * F(x) - F(y) - F(-y))$

OUTPUT:
By Sablik Lemma F has degree at most 3
$f(x) = 0$
$F(x) = A_1(x)$
$A_1(x)$ *is an arbitrary additive function.*
- 0.01 seconds -

Example 7.1.11. *(cf. Example 3 in [28] and Example 4.4.2 in Chapter 4)*

$$F(x+y) - F(x) - F(y) = xf(3y) + yf(3x)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x+y) - F(x) - F(y) - x * f(3 * y) - y * f(3 * x))$

OUTPUT:
By Sablik Lemma f has degree at most 3
$f(x) = B_1(x) + a_3x^2$
$F(x) = A_1(x) + 3xB_1(x) + 3a_3x^3$
where a_3 *is a real number.*
$A_1(x), B_1(x)$ *are arbitrary additive functions.*
- 0.05 seconds -

Example 7.1.12. *(cf. Example 4 in [28] and Example 4.4.3 in Chapter 4)*

$$F(x+y) - F(x) - F(y) = xf(y)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x+y) - F(x) - F(y) - x * f(y))$

OUTPUT:
By Sablik Lemma F has degree at most 2
$f(x) = a_2x$

$F(x) = A_1(x) + \frac{1}{2}a_2x^2$
where a_2 is a real number.
$A_1(x)$ is an arbitrary additive function.
- 0.02 seconds -

Example 7.1.13. *(cf. Example 5 in [28] and Example 4.4.4 in Chapter 4)*

$$F(x + y) - F(x) - F(y) = 3xf(2y) - 4yf(3x)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x + y) - F(x) - F(y) - 3 * x * f(2 * y) + 4 * y * f(3 * x))$

OUTPUT:
By Sablik Lemma f has degree at most 3
$f(x) = a_2x$
$F(x) = A_1(x) - 3a_2x^2$
where a_2 is a real number.
$A_1(x)$ is an arbitrary additive function.
- 0.06 seconds -

Example 7.1.14. *(cf. Example 6 in [28] and Example 4.4.5 in Chapter 4)*

$$F(x + y) - F(x) - F(y) = xf(3y) + yf(3x) + xf(y) + yf(x) + xf(2y) + yf(2x)$$

for all $x, y \in \mathbb{R}$.
INPUT:
$PSFE(F(x+y)-F(x)-F(y)-x*f(3*y)-y*f(3*x)-x*f(y)-y*f(x)-y*f(2*x)-x*f(2*y))$

OUTPUT:
By Sablik Lemma f has degree at most 6
$f(x) = B_1(x) + a_3x^2$
$F(x) = A_1(x) + 6xB_1(x) + \frac{14}{3}a_3x^3$
where a_3 is a real number.
$A_1(x), B_1(x)$ are arbitrary additive functions.
- 0.09 seconds -

Example 7.1.15. *Consider the square-norm-equation in [8] and [7]*

$$F(x + y) + F(x - y) = 2F(x) + 2F(y)$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x + y) + F(x - y) - 2 * F(x) - 2 * F(y))$

OUTPUT:
By Sablik Lemma F has degree at most 2
$f(x) = 0$
$F(x) = A^*_2(x)$
$A^*_2(x)$ is an arbitrary bi-additive function.
- 0.01 seconds -

Example 7.1.16. *Consider the polynomial equation for $n = 5$ in [8]*

$$F(x+6y) - 6F(x+5y) + 15F(x+4y) - 20F(x+3y) + 15F(x+2y) - 6F(x+y) + F(x) = 0$$

for all $x, y \in \mathbb{R}$.

INPUT:
$PSFE(F(x+6*y) - 6*F(x+5*y) + 15*F(x+4*y) - 20*F(x+3*y) + 15*F(x+2*y) - 6*F(x+y) + F(x))$

OUTPUT:
By Sablik Lemma F has degree at most 5
$f(x) = 0$
$F(x) = a_0 + A_1(x) + A_2^*(x) + A_3^*(x) + A_4^*(x) + A_5^*(x)$
where a_0 is a real number.
$A_1(x), A_2^*(x), A_3^*(x), A_4^*(x), A_5^*(x)$ *are arbitrary k-additive functions for $k \in \{1,2,3,4,5\}$.*
- 0.02 seconds -

Example 7.1.17. *In [32], P.K Sahoo arrived at the functional equation stemming from trapezoidal rule*

$$F(y) - F(x) = \frac{y-x}{6}\left[f(x) + 2f\left(\frac{2x+y}{3}\right) + 2f\left(\frac{x+2y}{3}\right) + f(y)\right]$$

for all $x, y \in \mathbb{R}$, where F is an antiderivative of f.

INPUT:
$PSFE(F(y) - F(x) - (y-x)*Rational(1/6)*(f(x) + 2*f((2*x+y)*Rational(1/3)) + 2*f((x+2*y)*Rational(1/3)) + f(y)))$

OUTPUT:
By Sablik Lemma f has degree at most 5
$f(x) = a_1 + a_2 x$
$F(x) = a_0 + a_1 x + \frac{1}{2}a_2 x^2$
where a_0, a_1, a_2 are real numbers.
- 0.02 seconds -

7.2 Conclusions and Future Research

A computer code developed in a python programming language has been presented for obtaining the polynomial solutions of the functional equation of type (7.0.1). The method's success can be attributed to the theoretical results obtained in Chapter 5 (cf. [26]). We note that the functional equation of type (7.0.1) consists of at most two unknown functions (in particular, at most one unknown function on either side of (7.0.1)), say F or f, where f is multiplied by a variable x and/or y. Therefore, we aim to extend the approach to consider a Pexider form of (7.0.1), that is, an equation with more than two unknown functions. Namely, an equation of the form

$$\sum_{i=1}^{n}\sum_{p=1}^{N}\gamma_{ip}F_p(a_{ip}x + b_{ip}y) = \sum_{j=1}^{m}\sum_{q=1}^{M}(\alpha_{jq}x + \beta_{jq}y)f_q(c_{jq}x + d_{jq}y), \tag{7.2.1}$$

$(F_p, f_q) : \mathbb{R} \to \mathbb{R}$, for every $x, y \in \mathbb{R}$, $\gamma_{ip}, \alpha_{jq}, \beta_{jq} \in \mathbb{R}, a_{ip}, b_{ip}, c_{jq}, d_{jq} \in \mathbb{Q}, i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}, n, m, N, M \in \mathbb{N}$ and its special forms. Below are examples of special forms of (7.2.1)

1) Cauchy additive functional equation, pexiderized

$$F_1(x+y) = F_2(x) + F_3(y)$$

for all $x, y \in \mathbb{R}$.

2) A generalization of the square-norm-equation in [8]

$$F_1(x+y) + F_2(x-y) = F_3(x) + F_4(y)$$

for all $x, y \in \mathbb{R}$.

3) The general class of linear functional equations considered in [8] and [7]

$$\sum_{i=1}^{n} \sum_{p=1}^{N} F_p(a_{ip}x + b_{ip}y) = 0$$

for all $x, y \in \mathbb{R}$, and $a_{ip}, b_{ip} \in \mathbb{Q}$.

4) P. K. Sahoo and T. Riedel, in [33] (see Chapter 3), considered the functional equations

$$F_1(x) - F_1(y) = (x-y)\left[f_1(x+y) + f_2(x) + f_2(y)\right]$$
$$xf_1(y) - yf_1(x) = (x-y)\left[f_2(x+y) + f_3(x) + f_3(y)\right]$$

for all $x, y \in \mathbb{R}$.

5) Equation (7.0.1)

$$\sum_{i=1}^{n} \gamma_{i1} F_1(a_{i1}x + b_{i1}y) = \sum_{j=1}^{m} (\alpha_{j1}x + \beta_{j1}y) f_1(c_{j1}x + d_{j1}y)$$

for all $x, y \in \mathbb{R}$, $\gamma_{i1}, \alpha_{j1}, \beta_{j1} \in \mathbb{R}$, and $a_{i1}, b_{i1}, c_{j1}, d_{j1} \in \mathbb{Q}$.

Next, we will consider non-polynomial solutions of the functional equation (7.2.1), and finally, we will study solutions of its inequalities. An example of such functional inequality was considered in [11], namely,

$$F_1(x+y) - F_1(x) - F_1(y) \geqslant xf_1(y) + yf_1(x)$$

for all $x, y \in \mathbb{R}$.

Bibliography

[1] **J. Aczél,** A mean value property of the derivative of quadratic polynomials - without mean values and derivatives, *Math. Magazine*, **58** (1985), 42–45.

[2] **J. Aczél, M. Kuczma,** On two mean value properties and functional equations associated with them, *Aequationes Math.*, **38** (1989), 216–235.

[3] **J. Almira, E. Shulman,** On polynomial functions on non-commutative groups, *J. Math. Anal. Appl.* **458** (2018), no. 1, 875–888.

[4] **C. Alsina, M. Sablik, J. Sikorska,** On a functional equation based upon a result of Gaspard Monge. *Journal of Geometry.* **85(1-2)**(2006), 1-6.

[5] **S. Baják, Z. Páles**, Computer aided solution of the invariance equation for two-variable Gini means. *Computers and Mathematics with Applications.* **58(2)** (2009), 334-340.

[6] **S. Baják, Z. Páles**, Computer aided solution of the invariance equation for two-variable Stolarsky means. *Applied mathematics and computation.* **216(11)**(2010), 3219-3227.

[7] **G. G. Borus and A. Gilányi,** Computer assisted solution of systems of two variable linear functional equations. *Aequationes Math.* **94(4)**(2020), 723-736.

[8] **A. Gilányi,** Solving linear functional equations with computer. *Math. Pannon.* **9(1)**(1998), 57-70.

[9] **Z. Daróczy, Gy. Maksa,** Functional equations on convex sets. *Acta Math. Hungar.* **68(3)** (1995), 187-195.

[10] **D.Z. Djoković,** A representation theorem for $(X_1 - 1)(X_2 - 1)\cdots(X_n - 1)$ and its applications. *Ann. Polon. Math.* **22** (1969/70), 189–198.

[11] **W. Fechner, E. Gselmann,** General and alien solutions of a functional equation and of a functional inequality. *Publ. Math. Debrecen* **80**(2012), no. 1-2, 143-154.

[12] **M. Fréchet,** Une définition fonctionelle des polynômes. *Nouv. Ann.* **49**(1909), 145-162.

[13] **J. Ger,** *On Sahoo-Riedel equations on a real interval.* Aequationes Math. **63** (2002), 168-179.

[14] **R. Ger,** *On extensions of polynomial functions.* Results Math. **26** (1994), 281-289.

[15] **R. Ger, M. Sablik,** Alien functional equations: a selective survey of results. In: *Developments in functional equations and related topics,* 107-147, Springer Optim. Appl., **124**, Springer, Cham, 2017.

[16] **A. Házy,** Solving linear two variable functional equations with computer. *Aequationes Math.* **67(1)** (2004), 47-62.

[17] **A. Házy,** Solving functional equations with computer. In *4th IEEE International Conference on Cognitive Infocommunications (CogInfoCom)*, 2013, 913-916. .

[18] **B. Koclęga-Kulpa, T. Szostok,** On some functional equations connected to Hadamard inequalities. *Aequationes Math.* **75(1)**(2008), 119-129.

[19] **B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz,** Some functional equations characterizing polynomials. *Tatra Mt. Math. Publ.* **44**(2009), 27-40.

[20] **B. Koclęga-Kulpa, T. Szostok, S. Wąsowicz,** On functional equations connected with quadrature rules. *Georgian Math. J.* **16**(2009), 725-736.

[21] **R. Larsen,** Functional Analysis. Marcel Dekker, New York, 1973.

[22] **A. Lisak, M. Sablik,** Trapezoidal rule revisited, *Bull. Inst. Math. Acad. Sin. (N.S.)*, **6** (2011), 347-360.

[23] **S. Mazur, W. Orlicz,** Grundlegende Eigenschaften der polynomischen Operationen, *Studia Math.*, **5** (1934), 50–68, 179–189.

[24] **M. Merkle,** Conditions for convexity of a derivative and some applications to the Gamma function. *Aequationes Math.* **55** (1998), 273-280.

[25] **T. Nadhomi, C. P. Okeke, M. Sablik and T. Szostok,** On a new class of functional equations satisfied by polynomial functions. *Aequationes Math.,* **95** (2021), 1095-1117.

[26] **C. P. Okeke,** Further results on a new class of functional equations satisfied by polynomial functions. *Results Math.* **78**, 96 (2023). https://doi.org/10.1007/s00025-023-01877-8

[27] **C. P. Okeke, W. Ogala and T. Nadhomi,** On symbolic computation of C.P. Okeke functional equations using python programming language. Submitted.

[28] **C. P. Okeke, M. Sablik,** Functional equation characterizing polynomial functions and an algorithm. *Results Math.* **77**, 125 (2022). Doi: 10.1007/s00025-022-01664-x.

[29] **C. P. Okeke, M. Sablik,** Characterizing locally polynomial functions on convex subsets of linear spaces. Submitted.

[30] **I. Pawlikowska,** *A method of solving functional equations on convex subsets of linear spaces.* Aequationes Math. **75** (2008), 1–28.

[31] **M. Sablik,** Taylor's theorem and functional equations, *Aequationes Math.*, **60** (2000), 258-267.

[32] **P.K. Sahoo,** On a functional equation associated with the trapezoidal rule. *Bull. Inst. Math. Acad. Sin. (N.S.)* **2**(2007), 67-82.

[33] **P.K. Sahoo and T. Riedel,** Mean value theorems and functional equations, World Scientific, Singapore–New Jersey–London–Hong Kong, 1998.

[34] **E. Shulman,** Each semipolynomial on a group is a polynomial. *J. Math. Anal. Appl.* **479(1)**(2019), 765–772.

[35] **L. Székelyhidi,** Convolution type functional equations on topological commutative groups, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[36] **T. Szostok,** Alienation of two general linear functional equations. *Aequationes Math.* **94(2)**(2020), 287-301.

[37] **G. Van der Lijn,** La définition fonctionnelle des polynômes dans les groupes abéliens, *Fund. Math.*, **33** (1939), 42-50.

[38] **J. Walorski,** On a problem connected with convexity of a derivative. *Aequationes Math.* **62** (2001), 262-264.

[39] **W. H. Wilson,** On a certain general class of functional equations, *Amer. J. Math.*, **40** (1918), 263-282.

[40] **K. Yosida,** Functional Analysis. Springer Verlag, Berlin-New York, 1980.