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„Uogólnienia nierówności Hadamarda oraz twierdzeń Frobeniusa i Dieudonné"

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Tytuł angielski: *Generalizations of the Hadamard inequality and Frobenius and Dieudonné theorems*

The dissertation has a synthetic form. It has two main chapters and a third additional one. The first chapter is focused on the matrix inequalities, and more especially to the Hadamard's inequality and the von Neumann inequality. The second chapter covers the issues connected with linear transformations of square matrices that preserve selected properties of the matrices. The main topic of the considerations is the classic theorems of well-known mathematicians: Jacques Hadamard, John von Neumann, Ferdinand Frobenius and Jean Dieudonné.

The subject of the first chapter is based on the Hadamard's inequality proved in 1893 and the von Neumann's Trace Inequality shown in 1937. The goal of the investigation was the strengthening of the classical Hadamard's inequality to the form of the optimized Hadamard's inequality. The results of these improvements have already been published in the articles [10] and [11]. The author of the dissertation is their main co-author. Nevertheless, the results of this work will be presented in the dissertation in a refreshed and expanded form. At the same time, the research also went towards the von Neumann's inequality for matrix traces. This resulted in new proof of the von Neumann's inequality in the case of the number of discussed matrices greater than two and refinement of the assumptions in the von Neumann's inequality for two Hermitian matrices. Interestingly, the Hadamard and von Neumann inequalities have a certain thread. One version of the von Neumann's inequality can be used to prove a stronger version of the Hadamard's inequality presented by the Soviet mathematician Mark Kreyn [6], which is also presented in the dissertation. This topic is still of great interest, as can be seen from, for example, a quite recent article [7] by Lin and Sinnamon from 2020.

The second chapter of the work is based on two theorems of Ferdinand Frobenius from 1897 and of Jean Dieudonné from 1949. The first theorem concerns linear transformations that preserve the matrix determinant, whereas the second theorem concerns linear transformations that preserve the set of nonsingular matrices. It can be said that these issues formed the basis of a new branch of matrix theory dealing with linear transformations preserving certain matrix properties, namely matrix ranks, eigenvalues and singular values [9], sets of orthogonal matrices [3], unitary matrices [8] and nilpotent matrices [4].

These considerations were often carried out at the expense of losing the arbitrariness of the structure over which the matrices are defined. Matrices have often been considered over the field of complex numbers or over any algebraically closed field. In the second chapter, the main considerations went in two directions. Firstly, the Frobenius theorem was successfully extended to all fields and Frobenius and Dieudonné theorems were extended to rings modulo k. The key here was that k is a power of prime for a ring modulo k. Secondly, the Dieudonné theorem for linear singular transformations was considered, obtaining a generalization of the theorem given by Botta in article [2] from algebraically closed fields to almost all fields (except for finite fields with the number of elements not greater than the matrix degree). The end of the second chapter of the dissertation focuses on detailed research and the coherence of the material over the form of a linear transformation that preserves matrix ranks. It turns out that the main role here is played by the fact whether the field over which the matrices are considered is algebraically closed. The topic of the second chapter is still relevant, as evidenced by the articles [1, 5, 12] from the last decade.

The third chapter of the dissertation consists of two appendices loosely related to the topic of the dissertation. The first one is about generating set of square matrices of degree 2 defined over the ring of integers and over the ring of integer number and over rings modulo k. This result was arose when writing the second chapter of the dissertation. The second supplement is connected with certain matrix transformations that preserve singularity and nonsingularity, rank or determinant of a matrix, and thus, completes the topic of the second chapter of the dissertation.

List of literature cited in the abstract

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