

Summary of the PhD thesis

Generalized schemas and rules of reasoning in fuzzy logic

Approximate reasoning is recently an important tool since it allows us to obtain meaningful conclusions from imprecise data. It has many applications in areas like: decision theory, risk analysis, fuzzy control and data mining. In classical logic, one of the most common used rule is modus ponens (note that in the thesis we will use a notion of a scheme of reasoning) which can be seen as:

$$\frac{A \rightarrow B \wedge A}{\therefore B}$$

Its generalized version can be described as follows:

$$\frac{\begin{array}{ll} \text{RULE:} & \text{IF } x \text{ is } A, \quad \text{THEN } y \text{ is } B. \\ \text{FACT:} & x \text{ is } A'. \end{array}}{\text{CONCLUSION:} \quad y \text{ is } B'}.$$

As we can see objects x, y have some properties A, B and A', B' . Usually, elements of a pair (A, A') are only slightly different and the same is required form a pair (B, B') . In approximate reasoning based on fuzzy sets these properties are represented by fuzzy sets. We are able to compute values of these fuzzy sets using some rules of inference - Zadeh's compositional rule of inference (*CRI*, see [9]) and Bandler-Kohout subproduct (*BKS*, see [2]). For the scheme of modus ponens, formulas that allow us to compute the conclusion's values are the following:

$$B'(y) := \sup_{x \in X} T(A'(x), I(A(x), B(y))), \quad y \in Y,$$

$$B'(y) := \inf_{x \in X} I(A'(x), T(A(x), B(y))), \quad y \in Y,$$

where T is a t-norm (or any other generalization of a classical conjunction) and I is a fuzzy implication (or a generalization of a classical implication).

One of the basic properties which is required for these rules is a property of interpolativity, which is nothing else but satisfying the classical version of modus ponens:

$$B(y) = \sup_{x \in X} T(A(x), I(A(x), B(y))), \quad y \in Y.$$

Now if we consider all possible values of fuzzy sets - all unit interval, we may obtain the following functional equations:

$$y = \sup_{x \in [0,1]} T(x, I(x, y)), \quad (\text{CRI-GMP})$$

$$y = \inf_{x \in [0,1]} I(x, T(x, y)), \quad (\text{BK-GMP})$$

which should be satisfied for every $y \in [0, 1]$.

In this thesis we examine generalizations of three other schemas of inference:

(i) hypothetical syllogism

$$\frac{A \rightarrow B \wedge B \rightarrow C}{\therefore A \rightarrow C}$$

(ii) modus tollens

$$\frac{A \rightarrow B \wedge \neg B}{\therefore \neg A}$$

(iii) law of reduction to absurdity

$$\frac{\neg A \rightarrow B \wedge \neg B}{\therefore A}$$

For these schemas of reasoning applied for two rules (CRI and BKS), we obtain the following equations:

$$I(x, y) = \sup_{z \in [0,1]} (T(I(x, z), I(z, y))), \quad x, y \in [0, 1], \quad (\text{CRI-GHS})$$

$$I_2(x, y) = \inf_{z \in [0,1]} I_1(T(x, z), T(z, y)), \quad x, y \in [0, 1], \quad (\text{BK-GHS})$$

$$N(x) = \sup_{y \in [0,1]} T(N(y), I(x, y)), \quad x \in [0, 1], \quad (\text{CRI-GMT})$$

$$N(x) = \inf_{y \in [0,1]} I(N(y), T(x, y)), \quad x \in [0, 1], \quad (\text{BK-GMT})$$

$$x = \sup_{y \in [0,1]} T(N(y), I(N(x), y)), \quad x \in [0, 1], \quad (\text{CRI-GRA})$$

$$x = \inf_{y \in [0,1]} I(N(y), T(N(x), y)), \quad x \in [0, 1], \quad (\text{BK-GRA})$$

where T is a t-norm, I, I_1, I_2 are fuzzy implications and N is a fuzzy negation. Moreover, we investigate functional inequalities which can be obtained from lattice operations in Boolean algebra and then from extension to some fuzzy connectives:

$$T(x, I(x, y)) \leq y, \quad x, y \in [0, 1] \quad (\text{MP})$$

$$T(I(x, z), I(z, y)) \leq I(x, y), \quad x, y, z \in [0, 1] \quad (\text{HS})$$

$$T(N(y), I(x, y)) \leq N(x), \quad x, y \in [0, 1] \quad (\text{MT})$$

$$T(N(y), I(N(x), y)) \leq x, \quad x, y \in [0, 1], \quad (\text{RA})$$

So far, there were investigations which concerned mainly generalized modus ponens [7], hypothetical syllogism (first research - at the end of last century, see [4]) and for other schemas - mostly for inequalities (HS), (MP), (MT), (RA) (see [8]).

This thesis is organized as follows. In Chapter 1 we recall the most important notions in the theory of fuzzy sets. In Chapter 2 we give some basic facts regarding approximate reasoning. In the main part of the thesis we consider functional equations and inequalities mentioned above when one function is given - usually a t-norm T (or a semicopula or any other generalization of a classical conjunction). Therefore we show some solutions for chosen families of fuzzy implications.

In Chapter 3 we focus on hypothetical syllogism - there are solutions for (CRI-GHS), (BK-GHS) and (HS), but also we present here some algebraic properties of a composition $\sup - T$.

Chapter 4 contains information regarding solutions of (CRI-GMP), (BK-GMP) and (MP). In Chapter 5 analogous facts for (CRI-GMT), (BK-GMT) and (MT) can be found. Solutions for (CRI-GRA), (BK-GRA) and (RA) have been described in Chapter 6.

In Chapter 7 we shortly give some remarks concerning some other possible functional equations that might be received when combining rules of inference and different fuzzy relations.

In last Chapter 8, we present a different method of reasoning - similarity based reasoning. Also we give some remarks for two main strategies in approximate reasoning - FITA (*First Infer Then Aggregate*) and FATI (*First Aggregate Then Infer*) with respect to some proven theorems.

Some of the results presented in the thesis have been already published in referred papers in proceedings [3, 6] which were obtained in collaboration with M. Baczyński and P. Helbin and [1, 5] - prepared with M. Baczyński.

References

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